(f) Symmetric, Skew-Symmetric, and Orthogonal Matrices

Symmetric Matrix

A real square matrix $A=\left[a_{ij}\right]$ is called symmetric if transposition leaves it unchanged,

$$A^T=A, \text{ thus } a_{ij}=a_{ji}.$$

The Eigen-values of symmetric matrix are always real.

Example: $A=\begin{bmatrix}-3 & 1 & 5\\ 1 & 0 & -2\\ 5 & -2 & 4\end{bmatrix}$

Skew-Symmetric Matrix

A real square matrix $A=\left[a_{ij}\right]$ is called skew-symmetric if transposition gives the negative of A,

$$A^T=-A, \text{ thus } a_{ij}=-a_{ji}.$$

Every skew-symmetric matrix has all main diagonal entries zero.

The Eigen-values of skew-symmetric matrix are pure imaginary or zero.

Example: $A=\begin{bmatrix}0 & 9 & -12\\ -9 & 0 & 20\\ 12 & -20 & 0\end{bmatrix}$

NOTE:

Any real square matrix A may be written as the sum of a symmetric matrix R and a skew-symmetric matrix S, where

$$R=\frac{1}{2}\left(A+A^T\right) \text{ and } S=\frac{1}{2}\left(A-A^T\right)\Rightarrow A=R+S$$

Example: $A=\begin{bmatrix}3 & -4 & -1\\ 6 & 0 & -1\\ -3 & 13 & -4\end{bmatrix}=R+S=\begin{bmatrix}3 & 1 & -2\\ 1 & 0 & 6\\ -2 & 6 & -4\end{bmatrix}+\begin{bmatrix}0 & -5 & 1\\ 5 & 0 & -7\\ -1 & 7 & 0\end{bmatrix}$

Orthogonal Matrix

A real square matrix $A=\left[a_{ij}\right]$ is called orthogonal if transposition gives the inverse of A,

$$A^T=A^{-1}$$

NOTE:

(i) A real square matrix is orthogonal if and only if its column vectors and also its row vectors form an **orthonormal system**, that is

$$a_i . a_j = a_i^T a_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Thus, $A^T A = I$

(ii) The determinate of an orthogonal matrix has the value $+1$ or -1.

(iii) The eigenvalues of an orthogonal matrix A are real or complex conjugate in pairs and have absolute value 1.

Example: $A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$

Its characteristic equation is $-\lambda^3 + \frac{2}{3}\lambda^2 + \frac{2}{3}\lambda - 1 = 0 \Rightarrow (\lambda + 1)\left(\lambda^2 - \frac{5}{3}\lambda + 1\right) = 0$

$\Rightarrow \lambda = -1, \quad \left(5 + i\sqrt{11}\right)/6, \quad \left(5 - i\sqrt{11}\right)/6$