(g) Hermitian, Skew-Hermitian, and Unitary Matrices (Complex Matrices)

The complex conjugate of a matrix A is formed by taking the complex conjugate of each element. Thus, $\overline{A} = \overline{a}_{ij}$.

For the conjugate transpose, we use the notation $\overline{A}^T = \overline{a}_{ij}$.

Example: $A = \begin{bmatrix} 3+4i & -5i \\ -7 & 6-2i \end{bmatrix}$, then $\overline{A}^T = \begin{bmatrix} 3-4i & -7 \\ 5i & 6+2i \end{bmatrix}$

Hermitian Matrix

A square matrix $A = \left[a_{ij}\right]$ is called Hermitian if

$$\overline{A}^T = A, \text{ that is } \overline{a}_{ji} = a_{ij}$$

- If A is Hermitian, the entries on the main diagonal must satisfy $\overline{a}_{jj} = a_{jj}$, that is they are real.
- If a Hermitian matrix is real, then $\overline{A}^T = A^T = A$. Hence a real Hermitian matrix is a symmetric matrix.
- The eigenvalues of a Hermitian matrix (and thus a symmetric matrix) are real.

Example: $A = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$. The eigenvalues are $9, 2$.

Skew- Hermitian Matrix

A square matrix $A = \left[a_{ij}\right]$ is called skew-Hermitian if

$$\overline{A}^T = -A, \text{ that is } \overline{a}_{ji} = -a_{ij}$$

- If A is skew-Hermitian, then entries on the main diagonal must satisfy $\overline{a}_{jj} = -a_{jj}$, hence a_{jj} must be pure imaginary or 0.
- If a skew-Hermitian matrix is real, then $\overline{A}^T = A^T = -A$. Hence a real skew-Hermitian matrix is a skew-symmetric matrix.
- The eigenvalues of a skew-Hermitian matrix (and thus a skew-symmetric matrix) are pure imaginary or 0.

Example: $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$.

The eigenvalues are $4i, -2i$.

Unitary Matrix

A square matrix $A=\left[a_{ij}\right]$ is called unitary if

$$\bar{A}^T = A^{-1}$$

- If a unitary matrix is real, then $\bar{A}^T = A^T = A^{-1}$. Hence a real unitary matrix is an orthogonal matrix.
- The eigenvalues of a unitary matrix (and thus an orthogonal matrix) have absolute value 1.

Example: $A=\begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$.

The eigenvalues are $\frac{1}{2}\sqrt{3}+\frac{1}{2}i,\ -\frac{1}{2}\sqrt{3}+\frac{1}{2}i$ and $\left|\pm\frac{1}{2}\sqrt{3}+\frac{1}{2}i\right|=1$