(h) Similarity of Matrices, Basis of Eigenvectors, and Diagonalisation

Eigenvectors of an $n\times n$ matrix A may (or may not) form a basis. If they do, we can use them for "diagonalizing" A, that is, for transforming it into diagonal form with the eigenvalues on the main diagonal.

Similarity of Matrices

An $n\times n$ matrix $\hat{A}$ is called similar to an $n\times n$ matrix A if $\hat{A}=P^{-1}AP$

For some (nonsingular) $n\times n$ matrix P. This transformation, which gives $\hat{A}$ from A, is called **similarity transformation.**

- If $\hat{A}$ is similar to A, then $\hat{A}$ has the same eigenvalues as A.
- If x is an eigenvector of A, then $y=P^{-1}x$ is an eigenvector of $\hat{A}$ corresponding to same eigenvalue.
- If $\lambda_1,\lambda_2,....\lambda_k$ be distinct eigenvalues of an $n\times n$ matrix. Then corresponding eigenvectors $x_1,x_2,....x_k$ form linearly independent set.

Basis of Eigenvectors

If an $n\times n$ matrix A has n distinct eigenvalues, then A has a basis of eigenvectors.

- A Hermitian, skew-Hermitian or unitary matrix has a basis of eigenvectors.
- A symmetric matrix has an orthonormal basis of eigenvectors.

Diagonalisation

If an $n\times n$ matrix A has a basis of eigenvectors, then

$$D=X^{-1}AX$$

is diagonal, with the eigenvalues of A as the entries on the main diagonal. Here X is the matrix with these eigenvectors as column vectors. Also

$$D^m=X^{-1}A^mX$$

A square matrix which is not diagonalizable is called *defective.*

Example: $A=\begin{bmatrix}5 & 4\\ 1 & 2\end{bmatrix}$ has eigenvalues $6,\ 1$. The corresponding eigenvectors are

$$\frac{1}{\sqrt{17}}\begin{bmatrix}4\\ 1\end{bmatrix}\text{ and }\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}.$$

Hence, $X = \begin{bmatrix} 4/\sqrt{17} & 1/\sqrt{2} \\ 1/\sqrt{17} & -1/\sqrt{2} \end{bmatrix}$ and $X^{-1} = \frac{1}{-5/\sqrt{34}} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{17} & 4/\sqrt{17} \end{bmatrix}$

Thus,

$$D = X^{-1}AX = -\frac{\sqrt{34}}{5} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{17} & 4/\sqrt{17} \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4/\sqrt{17} & 1/\sqrt{2} \\ 1/\sqrt{17} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$$