(i) Functional Matrices

If the matrix A is diagonalizable, then we can find a matrix X and a diagonal matrix D such that

$$D = X^{-1}AX \Rightarrow A = XDX^{-1} \text{ and } A^n = XD^nX^{-1}$$

Applying the power series definition to this decomposition, we find that $f(A)$ is defined by

$f(A) = I + \alpha A + \beta A^2 + \gamma A^3 + ...$ (α, β, γ are coefficient of Taylor Expansion)

$$\Rightarrow f(A) = XIX^{-1} + \alpha XDX^{-1} + \beta XD^2X^{-1} + \gamma XD^3X^{-1} + ... \quad \because A^n = XD^nX^{-1}$$

$$\Rightarrow f(A) = X\left[I + \alpha D + \beta D^2 + \gamma D^3 + ...\right]X^{-1}$$

$$\Rightarrow f(A) = Xf(D)X^{-1} \Rightarrow f(A) = X\begin{bmatrix} f(d_1) & 0 & \cdots & 0 \\ 0 & f(d_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(d_n) \end{bmatrix}X^{-1}$$

where $d_1, d_2, ...d_n$ denote the diagonal entries of D.

Note: If A is itself diagonal, then $f(A) = f(D)$

Note: Always try to express X as unitary matrix because its inverse is same. If X is not unitary matrix then we have to find its inverse.

Thus,

$$D = X^{-1}AX = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\because f(A) = Xf(D)X^{-1} \Rightarrow e^A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Rightarrow e^A = \begin{bmatrix} \frac{1}{\sqrt{2}}e & \frac{1}{\sqrt{2}}e^{-1} \\ \frac{1}{\sqrt{2}}e & -\frac{1}{\sqrt{2}}e^{-1} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(e+e^{-1}\right) & \frac{1}{2}\left(e-e^{-1}\right) \\ \frac{1}{2}\left(e-e^{-1}\right) & \frac{1}{2}\left(e+e^{-1}\right) \end{bmatrix}$$