3(a). Gradient

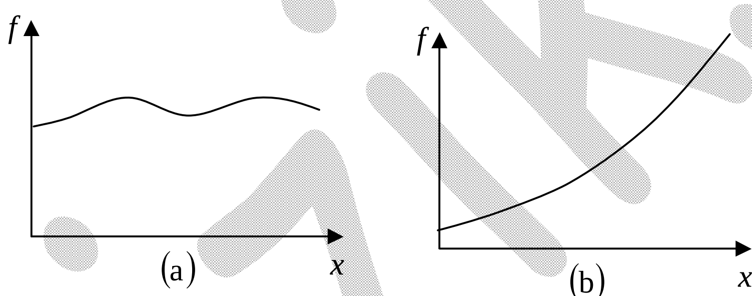
"Ordinary" Derivatives

Suppose we have a function of one variable: $f(x)$ then the derivative, df/dx tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx:

$$df = \left(\frac{df}{dx}\right)dx$$

In words: If we change x by an amount dx, then f changes by an amount df; the derivative is the proportionality factor. For example in figure (a), the function varies slowly with x, and the derivative is correspondingly small. In figure (b), f increases rapidly with x, and the derivative is large, as we move away from $x = 0$.

Geometrical Interpretation: The derivative df/dx is the slope of the graph of f versus x.



Suppose that we have a function of three variables-say, $V(x, y, z)$ in a

$$dV = \left(\frac{\partial V}{\partial x}\right)dx + \left(\frac{\partial V}{\partial y}\right)dy + \left(\frac{\partial V}{\partial z}\right)dz.$$

This tells us how V changes when we alter all three variables by the infinitesimal amounts dx, dy, dz. Notice that we do not require an infinite number of derivatives-three will suffice: the partial derivatives along each of the three coordinate directions.

Thus $$dV = \left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)\cdot\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right) = \left(\vec{\nabla}V\right)\cdot\left(d\vec{l}\right),$$

where $\vec{\nabla}V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$ is the gradient of V.

$\vec{\nabla}V$ is a vector quantity, with three components.

Geometrical Interpretation of the Gradient

Like any vector, the gradient has magnitude and direction. To determine its geometrical meaning, let's rewrite ; $dV = \vec{\nabla}V \cdot d\vec{l} = |\vec{\nabla}V||d\vec{l}|\cos\theta$

where θ is the angle between $\vec{\nabla}V$ and $d\vec{l}$. Now, if we fix the magnitude $|d\vec{l}|$ and search around in various directions (that is, vary θ), the maximum change in V evidently occurs when $\theta = 0$ (for then $\cos\theta = 1$). That is, for a fixed distance $|d\vec{l}|$, dV is greatest when one move in the same direction as $\vec{\nabla}V$. Thus: *The gradient $\vec{\nabla}V$ points in the direction of maximum increase of the function V.*

Moreover:

The magnitude $|\vec{\nabla}V|$ gives the slope (rate of increase) along this maximal direction.

Gradient in Spherical polar coordinates $V(r,\theta,\phi)$

$$\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

Gradient in cylindrical coordinates $V(r,\phi,z)$

$$\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}$$

Example: Find the unit vector normal to the curve $y = x^2$ at the point (2, 4, 1).

Solution: The equation of curve in the form of surface is given by $x^2 - y = 0$

A constant scalar function V on the surface is given by $V(x, y, z) = x^2 - y$

Taking the gradient

$$\vec{\nabla}V = \vec{\nabla}(x^2 - y) = \frac{\partial}{\partial x}(x^2 - y)\hat{x} + \frac{\partial}{\partial y}(x^2 - y)\hat{y} + \frac{\partial}{\partial z}(x^2 - y)\hat{z} = 2x\hat{x} - \hat{y}$$

The value of the gradient at point (2, 4, 1), $\vec{\nabla}V = 4\hat{x} - \hat{y}$

The unit vector, as required

$$\hat{n} = \pm\frac{4\hat{x} - \hat{y}}{|4\hat{x} - \hat{y}|} = \pm\frac{1}{\sqrt{17}}(4\hat{x} - \hat{y})$$