3(c). The Curl

From the definition of $\vec{\nabla}$ we construct the curl

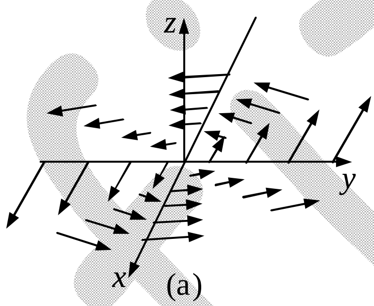
$$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ A_x & A_y & A_z\end{vmatrix}$$

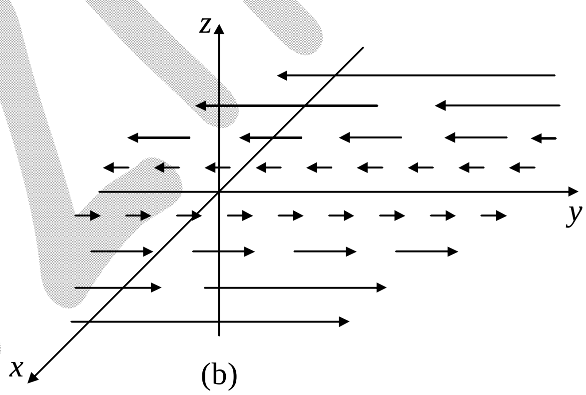
$$=\hat{x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\hat{y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\hat{z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

Notice that the curl of a vector function $\vec{A}$ is, like any cross product, a vector. (You cannot have the curl of a scalar; that's meaningless.)

Geometrical Interpretation

$\vec{\nabla}\times\vec{A}$ is a measure of how much the vector $\vec{A}$ "curls around" the point in question. Figure shown below have a substantial curl, pointing in the z-direction, as the natural right-hand rule would suggest.





Curl in Spherical polar coordinates $\vec{\nabla}\times\vec{A}=\frac{1}{r^2\sin\theta}\begin{pmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{pmatrix}$

Curl in cylindrical coordinates $\vec{\nabla}\times\vec{A}=\frac{1}{r}\begin{pmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z}\\ A_r & rA_\phi & A_z\end{pmatrix}$

Example: Suppose the function sketched in above figure are $\vec{A} = -y\hat{x} + x\hat{y}$ and $\vec{B} = x\hat{y}$. Calculate their curls.

Solution: $\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\hat{z}$ and $\vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = \hat{z}$

As expected, these curls point in the +z direction. (Incidentally, they both have zero divergence, as you might guess from the pictures: nothing is "spreading out"…. it just "curls around.")

Example: Given a vector function $\vec{A} = (x + c_1 z)\hat{x} + (c_2 x - 3z)\hat{y} + (x + c_3 y + c_4 z)\hat{z}$.

(a) Calculate the value of constants c_1, c_2, c_3 if $\vec{A}$ is irrotational.

(b) Determine the constant c_4 if $\vec{A}$ is also solenoidal.

(c) Determine the scalar potential function V, whose negative gradient equals $\vec{A}$.

Solution: If $\vec{A}$ is irrotational then, $\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x + c_1 z) & (c_2 x - 3z) & (x + c_3 y + c_4 z) \end{vmatrix} = 0$

$$\Rightarrow \vec{\nabla}\times\vec{A} = (c_3 + 3)\hat{x} - (1 - c_1)\hat{y} + (c_2 - 0)\hat{z} = 0 \Rightarrow c_1 = 1, c_2 = 0, c_3 = -3$$

(b) If $\vec{A}$ is solenoidal, $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z} = 1 + 0 + c_4 = 0 \Rightarrow c_4 = -1$

(c) $\vec{A} = -\vec{\nabla}V = -\dfrac{\partial V}{\partial x}\hat{x} - \dfrac{\partial V}{\partial y}\hat{y} - \dfrac{\partial V}{\partial z}\hat{z}$

$$\vec{A} = (x + z)\hat{x} + (-3z)\hat{y} + (x - 3y - z)\hat{z} \Rightarrow \frac{\partial V}{\partial x} = -x - z \Rightarrow V = -\frac{x^2}{2} - xz + f_1(y, z),$$

$$\frac{\partial V}{\partial y} = 3z \Rightarrow V = 3yz + f_2(x, z),\ \frac{\partial V}{\partial z} = -x + 3y + z \Rightarrow V = -xz + 3yz + \frac{z^2}{2} + f_3(x, y)$$

Examination of above expressions of V gives a general value of

$$V = -\frac{x^2}{2} - xz + 3yz + \frac{z^2}{2}$$

Example: Find the curl of the vector $\vec{A} = \left(e^{-r}/r\right)\hat{\theta}$

Solution: $\vec{A} = \left(e^{-r}/r\right)\hat{\theta} \Rightarrow A_r = 0,\ A_\theta = \left(e^{-r}/r\right),\ A_\phi = 0$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} = -\frac{e^{-r}}{r}\hat{\phi}$$

Example: Find the nature of the following fields by determining divergence and curl.

(i) $\vec{F}_1 = 30\hat{x} + 2xy\hat{y} + 5xz^2\hat{z}$

(ii) $\vec{F}_2 = \left(\frac{150}{r^2}\right)\hat{r} + 10\hat{\phi}$ (Cylindrical coordinates)

Solution:

(i) $\vec{F}_1 = 30\hat{x} + 2xy\hat{y} + 5xz^2\hat{z} \Rightarrow \vec{\nabla}\cdot\vec{F}_1 = \frac{\partial F_{1x}}{\partial x} + \frac{\partial F_{1y}}{\partial y} + \frac{\partial F_{1z}}{\partial z} = 2x(1+5z)$

Divergence exists, so the field is non-solenoidal.

$$\vec{\nabla}\times\vec{F}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 30 & 2xy & 5xz^2 \end{vmatrix} = -5z^2\hat{y} + 2y\hat{z}$$

.The field has a curl so it is rotational.

(ii) $\vec{F}_2 = \left(\frac{150}{r^2}\right)\hat{r} + 10\hat{\phi}$ in cylindrical coordinates.

In cylindrical coordinates, Divergence $\vec{\nabla}\cdot\vec{F}_2 = \frac{1}{r}\frac{\partial}{\partial r}\left(rF_{2r}\right) + \frac{1}{r}\frac{\partial F_{2\phi}}{\partial \phi} + \frac{\partial F_{2z}}{\partial z} = \frac{-150}{r^3}$

The field is non-solenoid.

$$\vec{\nabla}\times\vec{F}_2 = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \left(\frac{150}{r^2}\right) & 10r & 0 \end{vmatrix} = \frac{10}{r}\hat{z}$$

. $\vec{F}_2$ has non-zero curl so it is rotational.