3(d). Product Rules

The calculation of ordinary derivatives is facilitated by a number of general rules, such as

the sum rule: $\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx},$

the rule for multiplying by a constant: $\frac{d}{dx}(kf)=k\frac{df}{dx},$

the product rule: $\frac{d}{dx}(fg)=f\frac{dg}{dx}+g\frac{df}{dx},$

and the quotient rule: $\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{g\frac{df}{dx}-f\frac{dg}{dx}}{g^2}.$

Similar relations hold for the vector derivatives. Thus,

$$\vec{\nabla}(f+g)=\vec{\nabla}f+\vec{\nabla}g,\quad \vec{\nabla}\cdot(\vec{A}+\vec{B})=(\vec{\nabla}\cdot\vec{A})+(\vec{\nabla}\cdot\vec{B}),$$

$$\vec{\nabla}\times(\vec{A}+\vec{B})=(\vec{\nabla}\times\vec{A})+(\vec{\nabla}\times\vec{B}),$$

and

$$\vec{\nabla}(kf)=k\vec{\nabla}f,\quad \vec{\nabla}\cdot(k\vec{A})=k(\vec{\nabla}\cdot\vec{A}),\qquad \vec{\nabla}\times(k\vec{A})=k(\vec{\nabla}\times\vec{A}),$$

as you can check for yourself. The product rules are not quite so simple. There are two ways to construct a scalar as the product of two functions:

fg (product of two scalar functions),

$\vec{A}\cdot\vec{B}$ (Dot product of two vectors),

and two ways to make a vector:

$f\vec{A}$ (Scalar time's vector),

$\vec{A}\times\vec{B}$ (Cross product of two vectors),

Accordingly, there are six product rules,

Two for gradients

(i) $\vec{\nabla}(fg)=f\vec{\nabla}g+g\vec{\nabla}f,$

(ii) $\vec{\nabla}(\vec{A}\cdot\vec{B})=\vec{A}\times(\vec{\nabla}\times\vec{B})+\vec{B}\times(\vec{\nabla}\times\vec{A})+(\vec{A}\cdot\vec{\nabla})\vec{B}+(\vec{B}\cdot\vec{\nabla})\vec{A},$

Two for divergences

(iii) $\vec{\nabla}\cdot\left(f\vec{A}\right)=f\left(\vec{\nabla}\cdot\vec{A}\right)+\vec{A}\cdot\left(\vec{\nabla}f\right)$,

(iv) $\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)$,

And two for curls

(v) $\vec{\nabla}\times\left(f\vec{A}\right)=f\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\times\left(\vec{\nabla}f\right)$,

(vi) $\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A}-\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}+\vec{A}\left(\vec{\nabla}\cdot\vec{B}\right)-\vec{B}\left(\vec{\nabla}\cdot\vec{A}\right)$,

It is also possible to formulate three quotient rules:

$$\vec{\nabla}\left(\frac{f}{g}\right)=\frac{g\vec{\nabla}f-f\vec{\nabla}g}{g^2},\quad \vec{\nabla}\cdot\left(\frac{\vec{A}}{g}\right)=\frac{g\left(\vec{\nabla}\cdot\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}g\right)}{g^2},\quad \vec{\nabla}\times\left(\frac{\vec{A}}{g}\right)=\frac{g\left(\vec{\nabla}\times\vec{A}\right)+\vec{A}\times\left(\vec{\nabla}g\right)}{g^2}.$$