## 3(e). Second Derivatives

The gradient, the divergence, and the curl are the only first derivatives we can make with $\vec{\nabla}$; by applying $\vec{\nabla}$ twice we can construct five species of second derivatives. The gradient $\vec{\nabla}V$ is a vector, so we can take the *divergence* and *curl* of it:

**(1) Divergence of gradient:** $\vec{\nabla}\cdot\left(\vec{\nabla}V\right)$

$$\vec{\nabla}\cdot\left(\vec{\nabla}V\right)=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial V}{\partial x}\hat{x}+\frac{\partial V}{\partial y}\hat{y}+\frac{\partial V}{\partial z}\hat{z}\right)=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}.$$

This object, which we write $\nabla^2 V$ for short, is called the **Laplacian** of *V*. Notice that the Laplacian of a *scalar V* is a *scalar*.

**Laplacian in Spherical polar coordinates**

$$\nabla^2 V=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial\phi^2}\right)$$

**Laplacian in cylindrical coordinates**

$$\nabla^2 V=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}$$

Occasionally, we shall speak of the Laplacian of a *vector*, $\nabla^2\vec{A}$. By this we mean a *vector* quantity whose *x*-component is the Laplacian of $A_x$, and so on:

$$\nabla^2\vec{A}\equiv\left(\nabla^2 A_x\right)\hat{x}+\left(\nabla^2 A_y\right)\hat{y}+\left(\nabla^2 A_z\right)\hat{z}.$$

This is nothing more than a convenient extension of the meaning of $\nabla^2$.

**(2) Curl of gradient:** $\vec{\nabla}\times\left(\vec{\nabla}V\right)$

The divergence $\vec{\nabla}\cdot\vec{A}$ is a *scalar*-all we can do is taking its gradient.

The curl of a gradient is always *zero*: $\vec{\nabla}\times\left(\vec{\nabla}V\right)=0$.

**(3) Gradient of divergence:** $\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)$

The curl $\vec{\nabla}\times\vec{A}$ is a *vector*, so we can take its *divergence* and *curl*.

Notice that $\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)$ is not the same as the Laplacian of a vector:

$$\nabla^2\vec{A}=\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{A}\neq\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right).$$

**(4) Divergence of curl:** $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)$

The divergence of a curl, like the curl of a gradient, is *always zero*:

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0.$$

**(5) Curl of curl:** $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)$

As you can check from the definition of $\vec{\nabla}$:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)-\nabla^2\vec{A}.$$

So curl-of-curl gives nothing new; the first term is just number (3) and the second is the Laplacian (of a vector).

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