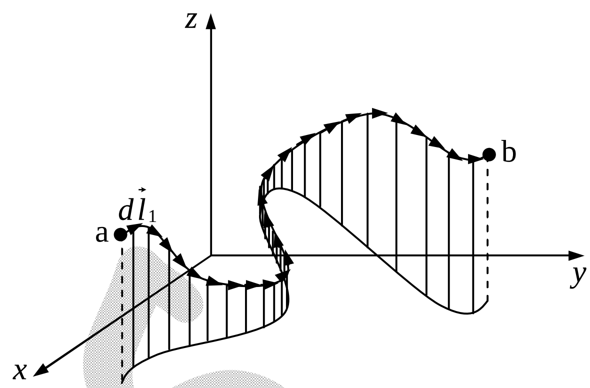
4(e). The Fundamental Theorem for Gradients

Suppose we have a scalar function of three variables $V(x, y, z)$. Starting at point a, we moves a small distance $d\vec{l}_1$. Then

$$dV = (\vec{\nabla} V)\cdot d\vec{l}_1 .$$



Now we move a little further, by an additional small displacement $d\vec{l}_2$; the incremental change in V will be $(\vec{\nabla} V)\cdot d\vec{l}_2$. In this manner, proceeding by infinitesimal steps, we make the journey to point b. At each step we compute the gradient of V (at that point) and dot it into the displacement $d\vec{l}$ …this gives us the change in V. Evidently the total change in V in going from a to b along the path selected is

$$\int_{a\,P}^{b} (\vec{\nabla} V)\cdot d\vec{l} = V(b) - V(a).$$

This is called the fundamental theorem for gradients; like the "ordinary" fundamental theorem, it says that the integral (here a line integral) of a derivative (here the gradient) is given by the value of the function at the boundaries (a and b).

Geometrical Interpretation

Suppose you wanted to determine the height of the Eiffel Tower. You could climb the stairs, using a ruler to measure the rise at each step, and adding them all up or you could place altimeters at the top and the bottom, and subtract the two readings; you should get the same answer either way (that's the fundamental theorem).

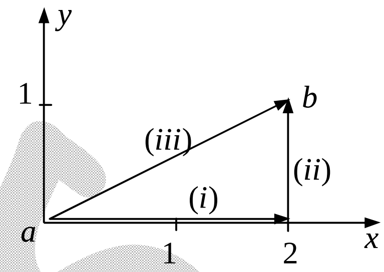
Corollary 1: $\int_a^b (\vec{\nabla} V)\cdot d\vec{l}$ is independent of path taken from a to b.

Corollary 2: $\oint (\vec{\nabla} V)\cdot d\vec{l} = 0$, since the beginning and end points are identical, and hence

$V(b) - V(a) = 0$.

Example: Let $V = xy^2$, and take point a to be the origin (0, 0, 0) and b the point (2, 1, 0). Check the fundamental theorem for gradients.

Solution: Although the integral is independent of path, we must pick a specific path in order to evaluate it. Let's go out along the x axis (step i) and then up (step ii). As always,



$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z},\ \vec{\nabla}V = y^2\hat{x} + 2xy\,\hat{y}$

(i) $y = 0;\ d\vec{l} = dx\,\hat{x},\ \vec{\nabla}V\cdot d\vec{l} = y^2 dx = 0,$ so $\int_i \vec{\nabla}V\cdot d\vec{l} = 0$

(ii) $x = 2;\ d\vec{l} = dy\,\hat{y},\ \vec{\nabla}V.d\vec{l} = 2xydy = 4y\,dy,$ so

$$\int_{ii}\vec{\nabla}V\cdot d\vec{l} = \int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2$$

Evidently the total line integral is 2.

This consistent with the fundamental theorem: T(b) – T(a) =2 – 0 = 2.

Calculate the same integral along path (iii) (the straight line from a to b):

(iii) $y = \frac{1}{2}x, dy = \frac{1}{2}dx, \vec{\nabla}V\cdot d\vec{l} = y^2 dx + 2xy\,dy = \frac{3}{4}x^2 dx,$ so

$\int_{iii}\vec{\nabla}V\cdot d\vec{l} = \int_0^2 \frac{3}{4}x^2\,dx = \frac{1}{4}x^3\Big|_0^2 = 2$. Thus the integral is independent of path.