4(g). The Fundamental Theorem for Curls

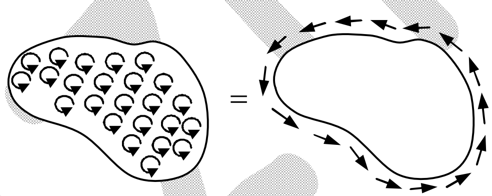
The fundamental theorem for curls, which goes by the special name of ***Stokes' theorem***, states that

$$\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \oint_P \vec{A}\cdot d\vec{l}$$

As always, the integral of a derivative (here, the curl) over a region (here, a patch of surface) is equal to the value of the function at the boundary (here, the perimeter of the patch). As in the case of the divergence theorem, the boundary term is itself an integral-specifically, a closed line integral.

Geometrical Interpretation:

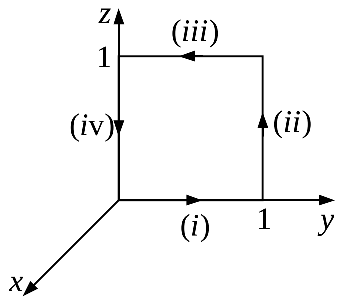
The integral of the curl over some surface (or, more precisely, the flux of the curl through that surface) represents the "total amount of swirl," and we can determine that swirl just as well by going around the edge and finding how much the flow is following the boundary (as shown in figure).



Corollary 1: $\int\left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint\left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a}=0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the right side of equation vanishes.

Example: Suppose $\vec{A} = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}$. Check Stokes' theorem for the square surface shown in figure.



Solution: Here

$\vec{\nabla} \times \vec{A} = (4z^2 - 2x)\hat{x} + 2z\,\hat{z}$ and $d\vec{a} = dy\,dz\,\hat{x}$

(In saying that $d\vec{a}$ points in the x direction, we are choosen to a counterclockwise line integral. We could as well write $d\vec{a} = -dy\,dz\,\hat{x}$, but then we have to go clockwise.) Since $x = 0$ for this surface,

$$\int (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \int_0^1 \int_0^1 4z^2\,dy\,dz = \frac{4}{3}$$

Now, what about the line integral? We must break this up into four segments:

(i) $x = 0, \quad z = 0, \quad \vec{A} \cdot d\vec{l} = 3y^2 dy, \qquad \int \vec{A} \cdot d\vec{l} = \int_0^1 3y^2 dy = 1,$

(ii) $x = 0, \quad y = 1, \quad \vec{A} \cdot d\vec{l} = 4z^2 dz, \int \vec{A} \cdot d\vec{l} = \int_0^1 4z^2 dz = \frac{4}{3},$

(iii) $x = 0, \quad z = 1, \quad \vec{A} \cdot d\vec{l} = 3y^2 dy, \qquad \int \vec{A} \cdot d\vec{l} = \int_1^0 3y^2 dy = -1,$

(iv) $x = 0, \quad y = 0, \quad \vec{A} \cdot d\vec{l} = 0, \qquad \int \vec{A} \cdot d\vec{l} = \int_1^0 0 dz = 0,$

So

$$\oint \vec{A} \cdot d\vec{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}.$$