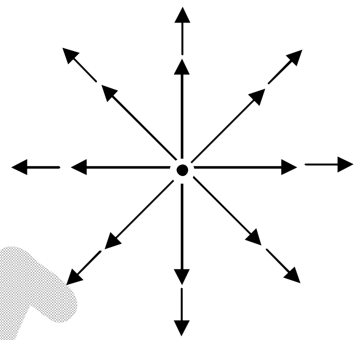
5(a). The Three-Dimensional Delta Function

The Divergence of $\hat{r}/r^2$

Consider the vector function

$$\vec{A} = \frac{1}{r^2}\hat{r}$$



At every location, $\vec{A}$ is directed radially outward. When we calculate the divergence we get precisely zero:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

The plot thickens if you apply the divergence theorem to this function. Suppose we integrate over a sphere of radius *R*, centered at the origin; the surface integral is

$$\oint \vec{A}\cdot d\vec{a} = \int\left(\frac{1}{R^2}\hat{r}\right)\cdot\left(R^2\sin\theta\, d\theta\, d\phi\,\hat{r}\right) = \left(\int_0^{\pi}\sin\theta\, d\theta\right)\left(\int_0^{2\pi}d\phi\right) = 4\pi$$

But the volume integral, $\int\left(\vec{\nabla}\cdot\vec{A}\right)d\tau$, is zero. Does this mean that the divergence theorem is false?

The source of the problem is the point $r = 0$, where $\vec{A}$ blows up. It is quite true that $\vec{\nabla}\cdot\vec{A} = 0$ everywhere except the origin, but right at the origin the situation is more complicated.

Notice that the surface integral is independent of *R*; if the divergence theorem is right (and it is), we should get $\int\left(\vec{\nabla}\cdot\vec{A}\right)d\tau = 4\pi$ for any sphere centered at the origin, no matter how small. Evidently the entire contribution must be coming from the point $r = 0$!

Thus, $\vec{\nabla}\cdot\vec{A}$ has the bizarre property that it vanishes everywhere except at one point, and yet its integral (over any volume containing that point) is 4π. No ordinary function behaves like that. (On the other hand, a physical example does come to mind: the density (mass per unit volume) of a point particle. It's zero except at the exact location of the particle, and yet its integral is finite namely, the mass of the particle.) What we have stumbled on is a mathematical object known to physicists as the ***Dirac delta function***.

It is an easy matter to generalize the delta function to three dimensions:

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

(As always, $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ is the position vector, extending from the origin to the point (x, y, z)). This three-dimensional delta function is zero everywhere except at (0, 0, 0), where it blows up. Its volume integral is 1

$$\int_{all\ space} \delta^3(\vec{r})d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dx\,dy\,dz = 1$$

and

$$\int_{all\ space} f(\vec{r})\delta^3(\vec{r}-\vec{a})d\tau = f(\vec{a})$$

Since the divergence of $\hat{r}/r^2$ is zero everywhere except at the origin, and yet its integral over any volume containing the origin is a constant (4π). These are precisely the defining conditions for the Dirac delta function; evidently

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$$

Example: Evaluate the integral $J = \int_v (r+1)\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau$ where v is a sphere of radius R centered at the origin.

Solution:

$$J = \int_v (r+1)\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau = \int_v (r+1)4\pi\delta^3(r)\,d\tau = 4\pi(0+1) = 4\pi$$