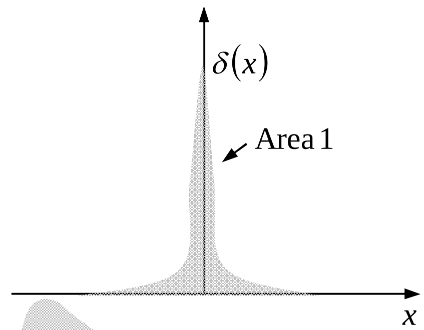
5(b). The One- Dimensional Dirac Delta Function

The one dimensional Dirac delta function, $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow "spike," with area 1 (as shown in figure).



That is to say:

$$\delta(x)=\begin{cases}0 & \text{if } x\neq 0\\ \infty, & \text{if } x=0\end{cases}$$

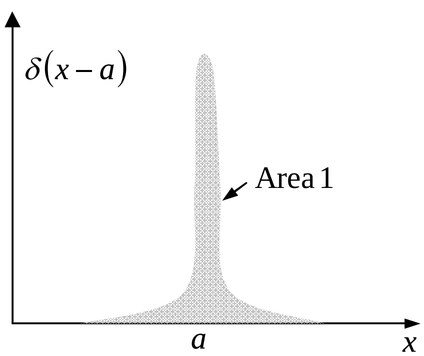
and $$\int_{-\infty}^{\infty}\delta(x)dx=1$$

If $f(x)$ is some "ordinary" function then the product $f(x)\delta(x)$ is zero everywhere except at $x=0$. It follows that

$$f(x)\delta(x)=f(0)\delta(x)$$

Of course, we can shift the spike from $x=0$ to some other point, $x=a$:

$$\delta(x-a)=\begin{cases}0, & \text{if } x\neq a\\ \infty, & \text{if } x=a\end{cases} \quad \text{with} \quad \int_{-\infty}^{\infty}\delta(\text{x-a})dx=1$$



also

$$f(x)\delta(x-a)=f(a)\delta(x-a)$$

and

$$\int_{-\infty}^{\infty}f(x)\delta(x-a)dx=f(a)$$

Example: Show that

$$\delta(kx)=\frac{1}{|k|}\delta(x),$$

where k is any (nonzero constant). (In particular $\delta(-x)=\delta(x)$.)

Solution: For an arbitrary test function $f(x)$, consider the integral $\int_{-\infty}^{\infty} f(x)\delta(kx)dx$.

Changing variables, we let $y=kx$, so that $x=y/k\,x$, and $dx=\frac{1}{k}dy$. If k is positive, the integration still runs from $-\infty$ to $+\infty$, but if k is negative, then $x=\infty$ implies $y=-\infty$, and vice versa, so the order of limit is reversed. Restoring the "proper" order costs a minus sign. Thus

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx=\pm\int_{-\infty}^{\infty} f(y/k)\delta(y)\frac{dy}{k}=\pm\frac{1}{k}f(0)=\frac{1}{|k|}f(0)$$

(The lower signs apply when k is negative, and we account for this neatly by putting absolute value bars around the final k, as indicated.) Under the integral sign, then, $\delta(kx)$ serves the same purpose as $(1/|k|)\delta(x)$:

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx=\int_{-\infty}^{\infty} f(x)\left[\frac{1}{|k|}\delta(x)\right]dx \Rightarrow \delta(kx)=(1/|k|)\delta(x)$$

Example: Evaluate the integral $I=\int_0^1 x^3\delta(x-2)\,dx$

Solution: Answer would be 0, because the spike would then be outside the domain of integration.

Example: Evaluate the integral $I=\int_0^3 x^3\delta(x-2)\,dx$

Solution:

The delta function picks out the value of x^3 at the point $x=2$ so the integral is $2^3=8$.

Example: Show that

$$x\frac{d}{dx}\left(\delta(x)\right)=-\delta(x) \qquad \text{and} \qquad \frac{d^n}{dx^n}\left(\delta(x)\right)=(-1)^n n!\frac{\delta(x)}{x^n}$$

Solution: $\int_{-\infty}^{\infty} f(x)\left[x\frac{d}{dx}\left(\delta(x)\right)\right]dx = xf(x)\delta(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{d}{dx}\left(xf(x)\right)\delta(x)dx$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\left[x\frac{d}{dx}\left(\delta(x)\right)\right]dx = -\int_{-\infty}^{\infty}\frac{d}{dx}\left(xf(x)\right)\delta(x)dx \qquad \because xf(x)\delta(x)\Big|_{-\infty}^{\infty}=0$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\left[x\frac{d}{dx}\left(\delta(x)\right)\right]dx = -\int_{-\infty}^{\infty}\left(x\frac{df}{dx}+f\right)\delta(x)dx = -f(0) = -\int_{-\infty}^{\infty} f(x)\delta(x)dx$$

$$\Rightarrow x\frac{d}{dx}\left(\delta(x)\right)=-\delta(x) \qquad \Rightarrow \frac{d}{dx}\left(\delta(x)\right)=-\frac{\delta(x)}{x}$$

$$\Rightarrow \frac{d^2}{dx^2}\left(\delta(x)\right)=\frac{d}{dx}\left(\frac{d}{dx}\delta(x)\right)=\frac{d}{dx}\left(-\frac{\delta(x)}{x}\right)=-\left(\delta(x)\frac{-1}{x^2}+\frac{1}{x}\frac{d}{dx}\delta(x)\right)$$

$$\Rightarrow \frac{d^2}{dx^2}\left(\delta(x)\right)=-\left(\delta(x)\frac{-1}{x^2}+\frac{1}{x}\left(\frac{-\delta(x)}{x}\right)\right)=2\frac{\delta(x)}{x^2}$$

Similarly $\frac{d^3}{dx^3}\left(\delta(x)\right)=-6\frac{\delta(x)}{x^3}$. Thus $\frac{d^n}{dx^n}\left(\delta(x)\right)=(-1)^n n!\frac{\delta(x)}{x^n}$

Example: Let $\theta(x)$ be the step function:

$$\theta(x)=\begin{cases}1, & \text{if } x>0\\ 0, & \text{if } x\le 0\end{cases}$$

Show that $\frac{d\theta}{dx}=\delta(x)$.

Solution:

$$\int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = f(x)\theta(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{df}{dx}\theta(x)dx = f(\infty)-\int_{0}^{\infty}\frac{df}{dx}dx = f(\infty)-\left[f(\infty)-f(0)\right]$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\frac{d\theta}{dx}dx = f(0)=\int_{-\infty}^{\infty} f(x)\delta(x)dx \Rightarrow \frac{d\theta}{dx}=\delta(x)$$