6. The Theory of Vector Fields

If the curl of a vector field ($\vec{F}$) vanishes (everywhere), then $\vec{F}$ can be written as the gradient of a scalar potential (V): $\vec{\nabla}\times\vec{F}=0 \Leftrightarrow \vec{F}=-\vec{\nabla}V$

(The minus sign is purely conventional.)

Theorem 1: Curl-less (or "irrotational") fields. The following conditions are equivalent (that is, $\vec{F}$ satisfies one if and only if it satisfies all the others):

(a) $\vec{\nabla}\times\vec{F}=0$ everywhere.

(b) $\int_a^b \vec{F}\cdot d\vec{l}$ is independent of path, for any given end points.

(c) $\oint \vec{F}\cdot d\vec{l}=0$ for any closed loop.

(d) $\vec{F}$ is the gradient of some scalar, $\vec{F}=-\vec{\nabla}V$.

The scalar potential is not unique-any constant can be added to V with impunity, since this will not affect its gradient.

If the divergence of a vector field ($\vec{F}$) vanishes (everywhere), then $\vec{F}$ can be expressed as the curl of a vector potential $(\vec{A})$:

$$\vec{\nabla}\cdot\vec{F}=0 \Leftrightarrow \vec{F}=\vec{\nabla}\times\vec{A}$$

That's the main conclusion of the following theorem:

Theorem 2: Divergence-less (or "solenoidal") fields. The following conditions are equivalent:

(a) $\vec{\nabla}\cdot\vec{F}=0$ everywhere.

(b) $\int \vec{F}\cdot d\vec{a}$ is independent of surface, for any given boundary line.

(c) $\oint \vec{F}\cdot d\vec{a}=0$ for any closed surface.

(d) $\vec{F}$ is the curl of some vector, $\vec{F}=\vec{\nabla}\times\vec{A}$.

The vector potential is not unique-the gradient of any scalar function can be added to $\vec{A}$ without affecting the curl, since the curl of a gradient is zero.