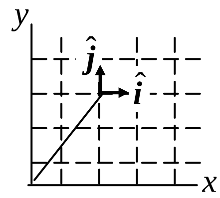
(a) Position Vector in Two Dimensional Plane

1.1 Two dimensional motion in Cartesian coordinate

The position vector in two dimension (x, y) plane is given by $\vec{r} = x\hat{i} + y\hat{j}$ where $\hat{i}, \hat{j}$ are unit vector in x and y direction respectively.



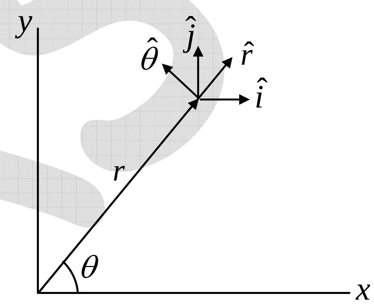
The base unit vector $\hat{i}$ and $\hat{j}$ are not vary with position as shown in figure.

The velocity is given by $\vec{v} = \dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j}$ and acceleration is given by $\vec{a} = \ddot{\vec{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$

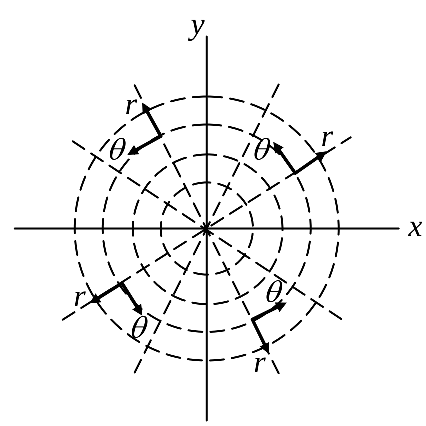
Newton's law can be written as $m\ddot{x} = F_x$ and $m\ddot{y} = F_y$.

1.2 Two dimensional motion in polar coordinate.

Two dimensional system also can be represent in polar coordinate with variable (r, θ) with transformation rule $x = r\cos\theta$ and $y = r\sin\theta$ where $r = \sqrt{x^2 + y^2}$ where r identified as magnitude of vector and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and θ is angle measured from x axis in anti clock wise direction as shown in figure.



In polar coordinate system $\hat{r}$ and $\hat{\theta}$ are unit vector in radial direction and tangential direction of trajectory. One can see from the figure the $\hat{r}$ and $\hat{\theta}$ are vary with position, where $|\hat{\mathbf{r}}| = 1, |\hat{\theta}| = 1$ and $\hat{r}.\hat{\theta} = 0$ conclude they are orthogonal in nature.



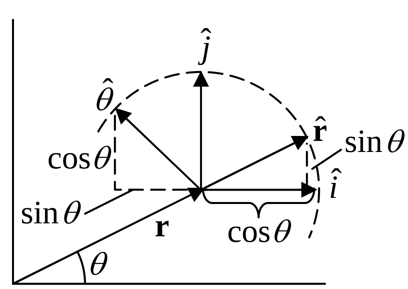
The unit vector $\hat{r}$ and $\hat{\theta}$ can be written in basis of unit vector $\hat{i}$ and $\hat{j}$.

The unit vectors $\hat{i}, \hat{j}$ and $\hat{r}, \hat{\theta}$ at a point in the xy-plane. We see that the orthogonality of $\hat{\mathbf{r}}$ and $\hat{\theta}$ plus the fact that they are unit vectors,

$|\hat{\mathbf{r}}| = 1, |\hat{\theta}| = 1$,

$\hat{\mathbf{r}} = \hat{i}\cos\theta + \hat{j}\sin\theta$ and

$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$ which is shown



The transformation can be shown by rotational Matrix

$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\end{bmatrix}$$

1.3 Time evolution of $\hat{r}$ and $\hat{\theta}$

$$\hat{\mathbf{r}}=\hat{i}\cos\theta+\hat{j}\sin\theta\Rightarrow\frac{d\hat{r}}{dt}=-\hat{i}\sin\theta\dot{\theta}+\hat{j}\cos\theta\dot{\theta}\Rightarrow\frac{d\hat{r}}{dt}=\dot{\theta}\hat{\theta}$$

$$\hat{\theta}=-\hat{i}\sin\theta+\hat{j}\cos\theta\Rightarrow\frac{d\hat{\theta}}{dt}=-\hat{i}\cos\theta\dot{\theta}-\hat{j}\sin\theta\dot{\theta}\Rightarrow\frac{d\hat{\theta}}{dt}=-\hat{r}\dot{\theta}$$

One can easily see unit vector $\hat{r}$ and $\hat{\theta}$ are vary with time .