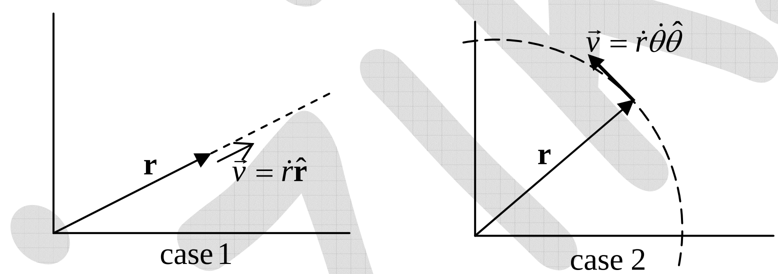
(b) Velocity and Acceleration in Polar Coordinate

1.1 The Position Vector in Polar Coordinate

$$\vec{r} = x\hat{i} + y\hat{j},\ \vec{r} = |r|\left[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right] \Rightarrow \vec{r} = |r|\hat{r}$$

$\mathbf{r} = r\hat{\mathbf{r}}$ is sometimes confusing, because the equation as written seems to make no reference to the angle θ. We know that two parameters needed to specify a position in two dimensional space (in Cartesian coordinates they are x and y), but the equation $\mathbf{r} = r\hat{\mathbf{r}}$ seems to contain only the quantity r. The answer is that $\hat{\mathbf{r}}$ is not a fixed vector and we need to know the value of θ to tell how $\hat{\mathbf{r}}$ is origin. Although θ does not occur explicitly in $r\hat{\mathbf{r}}$, its value must be known to fix the direction of $\hat{\mathbf{r}}$. This would be apparent if we wrote $\mathbf{r} = r\hat{\mathbf{r}}(\theta)$ to emphasize the dependence of $\hat{\mathbf{r}}$ on θ. However, by common conversation $\hat{\mathbf{r}}$ is understood to stand for $\hat{\mathbf{r}}(\theta)$.

1.2 Velocity Vector in Polar Coordinate



$$\vec{v} = \frac{d(r\hat{r})}{dt} = \frac{dr}{dt}.\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \Rightarrow \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

where $\dot{r}$ is radial velocity in $\hat{r}$ direction and $r\dot{\theta}$ is tangential velocity in $\hat{\theta}$ direction as shown in figure and the magnitude to velocity vector $|v| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$

1.3 Acceleration Vector in Polar Coordinate

$$\frac{d\vec{v}}{dt} = \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\dot{\theta}\hat{\theta} + r\frac{d\dot{\theta}}{dt}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$\frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}(-\dot{\theta})\hat{r}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta} \Rightarrow \vec{a} = a_r\hat{r} + a_\theta\hat{\theta}$$

$a_r = \ddot{r} - r\dot{\theta}^2$ is radial acceleration and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ is tangential acceleration .

So Newton's law in polar coordinate can be written as

$F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$ where F_r is external force in radial direction .

$F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ where F_θ is external force in tangential direction.

Example: In planar polar co-ordinates, an object's position at time t is given as $(r,\theta)=\left(e^t m,\sqrt{8}\,t \text{ rad}\right)$

(a) Find radial velocity tangential velocity and speed of particle at $t=0$

(b) Find radial acceleration tangential acceleration and magnitude of acceleration at $t=0$

Solution: (a) $(r,\theta)=\left(e^t m,\sqrt{8}\,t \text{ rad}\right)$

$r=e^t, \theta=\sqrt{8}\,t \text{ rad}$ $\quad v_r=\dot{r}\Rightarrow v_r=e^t$ at $t=0$ $v_r=1m/\sec$

And $v_\theta=r\dot{\theta}\Rightarrow e^t.\sqrt{8}$ at $t=0$ $\quad v_\theta=r\dot{\theta}\Rightarrow e^t.\sqrt{8}=\sqrt{8}$

Speed of particle is given by $|v|=\sqrt{v_r^2+v_\theta^2}=\sqrt{(1)^2+\left(\sqrt{8}\right)^2}=\sqrt{9}=3m/\sec$

(b) Magnitude of acceleration $a_r=\ddot{r}-\dot{\theta}^2 r\Rightarrow e^t-8\times e^t$ at $t=0$ $,1-8\times1=-7m/\sec^2$

$$a_\theta=r\ddot{\theta}+2\dot{r}\dot{\theta}\Rightarrow e^t\times0+2e^t\sqrt{8}=2\sqrt{8}$$

Magnitude of acceleration is $|a|=\sqrt{(-7)^2+4\times8}=\sqrt{49+32}=\sqrt{81}=9m/\sec^2$