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GATE Physics-2024

Solution-Mathematical Physics

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Q.11 – Q.35 Carry ONE mark Each

Q30. Consider a volume integral $I = \int_V \nabla^2 \left(\frac{1}{r} \right) dV$

over a volume V , where $r = \sqrt{x^2 + y^2 + z^2}$. Which of the following statement is/are correct?

- (A) $I = -4\pi$, if $r = 0$ is inside the volume V
- (B) Integrand vanishes for $r \neq 0$
- (C) $I = 0$, if $r = 0$ is not inside the volume V
- (D) Integrand diverges as $r \rightarrow \infty$

Ans: (A), (B), (C)

Solution:

$$\nabla^2 \left(\frac{1}{r} \right) = \vec{\nabla} \cdot \left(\vec{\nabla} \frac{1}{r} \right) = \vec{\nabla} \cdot \left(\frac{-\hat{r}}{r^2} \right) = -4\pi \delta^3(\vec{r})$$

$$I = \int_V \nabla^2 \left(\frac{1}{r} \right) dV = - \int_V 4\pi \delta^3(\vec{r}) dV; \quad \delta^3(\vec{r}) = \begin{cases} \infty, & \vec{r} = 0 \\ 0, & \vec{r} \neq 0 \end{cases}$$

$I = -4\pi$ if $r = 0$ is inside volume V .

$I = 0$ if $r = 0$ is not inside V .

$I = 0$ if $r \neq 0$

Q31. The complex function $e^{-\left(\frac{2}{z-1}\right)}$ has _____

- (A) a simple pole at $z = 1$
- (B) an essential singularity at $z = 1$
- (C) a residue equal to -2 at $z = 1$
- (D) a branch point at $z = 1$

Ans: (B), (C)

Solution:

$$f(z) = e^{-\left(\frac{2}{z-1}\right)} = 1 - \frac{2}{z-1} + \frac{4}{(z-1)^2} \cdot \frac{1}{2!} - \frac{8}{(z-1)^3} \cdot \frac{1}{3!} + \dots$$

Since there is infinite negative powers of $(z-1)$ so it has essential singularity at $z = 1$

Residue at $z = 1$ is $= (-2)$

Q.36 – Q.65 Carry TWO marks Each

Q43. Consider two matrices: $P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Which of the following statement is/are true?

- (A) P and Q have same set of eigenvalues
- (B) P and Q commute with each other
- (C) P and Q have different sets of linearly independent eigenvectors
- (D) P is diagonalizable

Ans: (A), (B), (C)

Solution:

Characteristic equation: $|P - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1, 1$

and $|Q - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1, 1$

$\therefore Q = I$ so $PQ = P = QP$

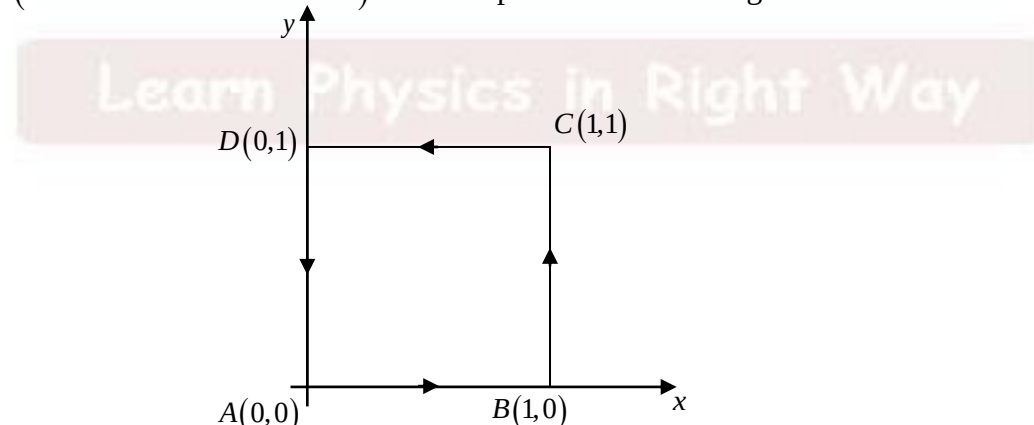
$\therefore PX = \lambda X \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1 + 2x_2 = x_1 \Rightarrow x_2 = 0$

$\Rightarrow X_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $X_1^T X_2 = 0 \Rightarrow \begin{bmatrix} x_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1^2 = 0 \Rightarrow x_1 = 0, \quad X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore Q = I$ so any non-zero vector is an eigenvector of Q having eigenvalue.

Q45. Consider a vector field $\vec{F} = (2xz + 3y^2)\hat{y} + 4yz^2\hat{z}$. The closed path

($\Gamma: A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$) in $z = 0$ plane is shown in figure.



$\oint_{\Gamma} \vec{F} \cdot d\vec{l}$ denotes the line integral of $\vec{F}$ along the closed path Γ . Which of the following option is/are true?

(A) $\oint \vec{F} \cdot d\vec{l} = 0$

(B) $\vec{F}$ is non-conservative

(C) $\vec{\nabla} \cdot \vec{F} = 0$

(D) $\vec{F}$ can be written as the gradient of a scalar field**Ans: (A), (B)****Solution:**

$$\vec{F} = (2xz + 3y^2)\hat{y} + 4yz^2\hat{z}$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(2xz + 3y^2) + \frac{\partial}{\partial z}(4yz^2) = 6y + 8yz = 6y \neq 0 \quad \because z = 0$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2xz + 3y^2 & 4yz^2 \end{vmatrix} = \hat{x}[4z^2 - 2x] - \hat{y}[0 - 0] + \hat{z}[2z - 0] = (4z^2 - 2x)\hat{x} + 2z\hat{z}$$

$$\Rightarrow \vec{\nabla} \times \vec{F} = -2x\hat{x} \neq 0 \quad \because z = 0 \quad [\vec{F} \text{ is non-conservative}]$$

$$\text{Now } \oint \vec{F} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{F}) \cdot d\vec{a} = \iint (-2x\hat{x}) \cdot (dxdy\hat{z}) = 0$$

Q51. A^α and B_β ($\alpha, \beta = 1, 2, 3, \dots, n$) are contravariant and covariant vectors, respectively. By convention, any repeated indices are summed over. Which of the following expression is/are tensors?

(A) $A^\alpha B_\beta$

(B) $\frac{A^\alpha B_\beta}{A^\alpha B_\alpha}$

(C) $\frac{A^\alpha}{B_\beta}$

(D) $A^\alpha + B_\beta$

Ans: (A), (B)

Q56. The Fourier transform and its inverse transform are respectively defined as

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{i\omega x} dx \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{f}(\omega) e^{-i\omega x} d\omega.$$

Consider two functions f and g . Another function $f * g$ is defined as

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) g(x - y) dy$$

Which of the following relation is/are true?

Note: Tilde ($\sim$) denote the Fourier transform.

(A) $f * g = g * f$

(B) $\widetilde{f * g} = \widetilde{g * f}$

(C) $\widetilde{f * g} = \widetilde{f} \widetilde{g}$

(D) $\widetilde{f * g} = \widetilde{f} \widetilde{g}$

Ans: (A), (B), (D)