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An Institute of NET-JRF, IIT-JAM, GATE, JEST, TIFR & CUET in Physics & Physical Sciences

MATHEMATICAL PHYSICS

(NET/JRF, GATE, JEST, TIFR)

CONTENTS
MATHEMATICAL PHYSICS

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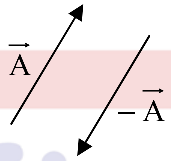
CHAPTER 1
VECTOR ANALYSIS

1.1 Vector Algebra

Vector quantities have both *direction* as well as *magnitude* such as velocity, acceleration, force and momentum etc. We will use $\vec{A}$ for any general vector and its magnitude by $|\vec{A}|$. In diagrams vectors are denoted by arrows: the length of the arrow is proportional to the magnitude of the vector, and the arrowhead indicates its direction. Minus $\vec{A}\,(-\vec{A})$ is a vector with the same magnitude as $\vec{A}$ but of opposite direction.



1.1.1 Vector Operations

We define four vector operations: addition and three kinds of multiplication.

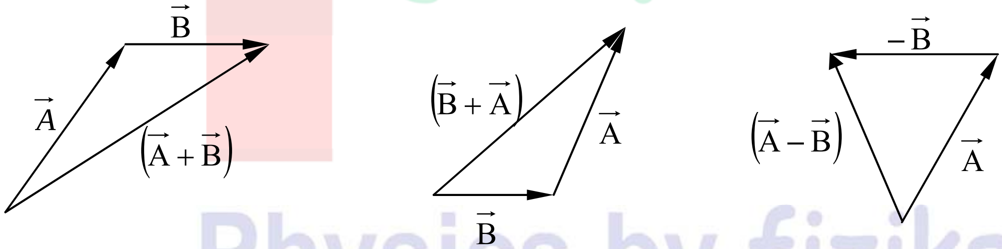
(i) Addition of two vectors

Place the tail of $\vec{B}$ at the head of $\vec{A}$; the sum, $\vec{A}+\vec{B}$, is the vector from the tail of $\vec{A}$ to the head of $\vec{B}$.

Addition is *commutative*: $\vec{A}+\vec{B}=\vec{B}+\vec{A}$

Addition is associative: $\left(\vec{A}+\vec{B}\right)+\vec{C}=\vec{A}+\left(\vec{B}+\vec{C}\right)$

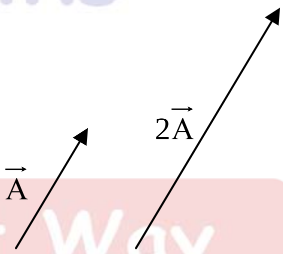
To subtract a vector, add its opposite: $\vec{A}-\vec{B}=\vec{A}+\left(-\vec{B}\right)$



(ii) Multiplication by scalar

Multiplication of a vector by a positive scalar *a*, multiplies the *magnitude* but leaves the direction unchanged. (If *a* is negative, the direction is reversed.) Scalar multiplication is distributive:

$$a\left(\vec{A}+\vec{B}\right)=a\vec{A}+a\vec{B}$$



(iii) Dot product of two vectors

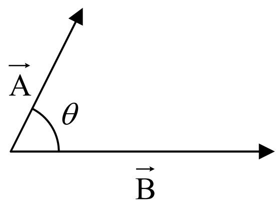
The dot product of two vectors is define by

$$\vec{A}.\vec{B}=AB\cos\theta$$

where θ is the angle they form when placed tail to tail. Note that $\vec{A}.\vec{B}$ is itself a scalar. The dot product is commutative,

$$\vec{A}.\vec{B}=\vec{B}.\vec{A}$$

and distributive, $\quad \vec{A}.\left(\vec{B}+\vec{C}\right)=\vec{A}.\vec{B}+\vec{A}.\vec{C}$



Geometrically $\vec{A}.\vec{B}$ is the product of A times the projection of $\vec{B}$ along $\vec{A}$ (or the product of B times the projection of $\vec{A}$ along $\vec{B}$).

If the two vectors are parallel, $\vec{A}.\vec{B}=AB$

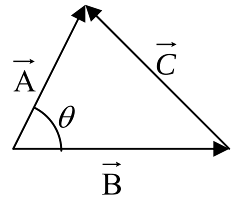
If two vectors are perpendicular, then $\vec{A}.\vec{B}=0$

Law of cosines

Let $\vec{C}=\vec{A}-\vec{B}$ and then calculate dot product of $\vec{C}$ with itself.

$\vec{C}.\vec{C}=\left(\vec{A}-\vec{B}\right).\left(\vec{A}-\vec{B}\right)=\vec{A}.\vec{A}-\vec{A}.\vec{B}-\vec{B}.\vec{A}+\vec{B}.\vec{B}$

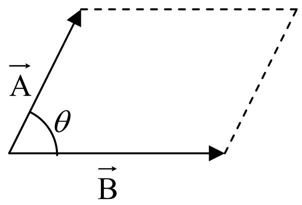
$C^2=A^2+B^2-2AB\cos\theta$



(iv) Cross product of two vectors

The cross product of two vectors is define by

$$\vec{A}\times\vec{B}=AB\sin\theta\hat{n}$$



where $\hat{n}$ is a unit vector (vector of length 1) pointing perpendicular to the plane of $\vec{A}$ and $\vec{B}$.Of course there are two directions perpendicular to any plane "in" and "out."

The ambiguity is resolved by the **right-hand rule:**

Let your fingers point in the direction of first vector and curl around (via the smaller angle) toward the second; then your thumb indicates the direction of $\hat{n}$. (In figure $\vec{A}\times\vec{B}$ points into the page; $\vec{B}\times\vec{A}$ points out of the page)

The cross product is distributive,

$$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\left(\vec{A}\times\vec{B}\right)+\left(\vec{A}\times\vec{C}\right)$$

but not commutative.

In fact, $\left(\vec{B}\times\vec{A}\right)=-\left(\vec{A}\times\vec{B}\right)$.

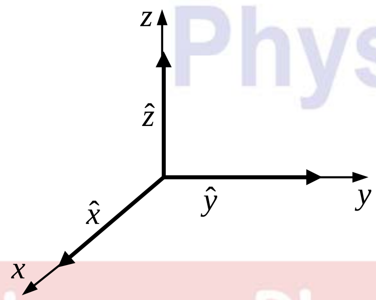
Geometrically, $\left|\vec{A}\times\vec{B}\right|$ is the area of the parallelogram generated by $\vec{A}$ and $\vec{B}$. If two vectors are parallel, their cross product is zero.

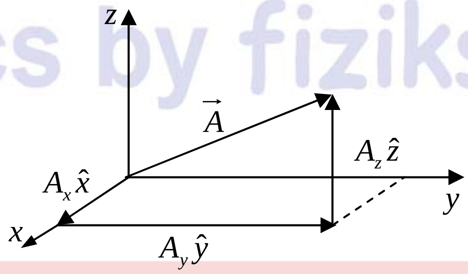
In particular $\vec{A}\times\vec{A}=0$ for any vector $\vec{A}$

1.1.2 Vector Algebra: Component Form

Let $\hat{x}$, $\hat{y}$ and $\hat{z}$ be unit vectors parallel to the x, y and z axis, respectively. An arbitrary vector $\vec{A}$ can be expanded in terms of these basis vectors

$$\vec{A}=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$$





The numbers A_x, A_y, and A_z are called component of $\vec{A}$; geometrically, they are the projections of $\vec{A}$ along the three coordinate axes.

(i) Rule: To add vectors, add like components.

$$\vec{A}+\vec{B}=\left(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\right)+\left(B_x\hat{x}+B_y\hat{y}+B_z\hat{z}\right)=\left(A_x+B_x\right)\hat{x}+\left(A_y+B_y\right)\hat{y}+\left(A_z+B_z\right)\hat{z}$$

(ii) Rule: To multiply by a scalar, multiply each component.

$$\vec{A}=\left(aA_x\right)\hat{x}+\left(aA_y\right)\hat{y}+\left(aA_z\right)\hat{z}$$

Because $\hat{x}$, $\hat{y}$ and $\hat{z}$ are mutually perpendicular unit vectors

$$\hat{x}.\hat{x}=\hat{y}.\hat{y}=\hat{z}.\hat{z}=1;\ \hat{x}.\hat{y}=\hat{x}.\hat{z}=\hat{y}.\hat{z}=0$$

Accordingly, $\vec{A}.\vec{B}=\left(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\right).\left(B_x\hat{x}+B_y\hat{y}+B_z\hat{z}\right)=A_xB_x+A_yB_y+A_zB_z$

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(iii) Rule: To calculate the dot product, multiply like components, and add.

In particular, $\vec{A}.\vec{A} = A_x^2 + A_y^2 + A_z^2 \Rightarrow A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Similarly, $\hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0,$

$\hat{x}\times\hat{y} = -\hat{y}\times\hat{x} = \hat{z}$

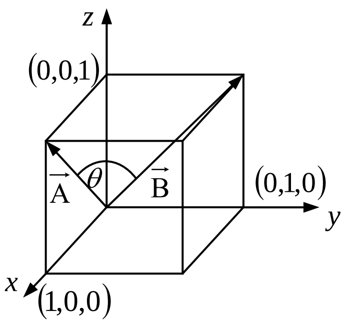
$\hat{y}\times\hat{z} = -\hat{z}\times\hat{y} = \hat{x}$

$\hat{z}\times\hat{x} = -\hat{x}\times\hat{z} = \hat{y}$

(iv) Rule: To calculate the cross product, form the determinant whose first row is $\hat{x}, \hat{y}, \hat{z}$, whose second row is $\vec{A}$ (in component form), and whose third row is $\vec{B}$.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \left(A_yB_z - A_zB_y\right)\hat{x} + \left(A_zB_x - A_xB_z\right)\hat{y} + \left(A_xB_y - A_yB_x\right)\hat{z}$$

Example: Find the angle between the face diagonals of a cube.



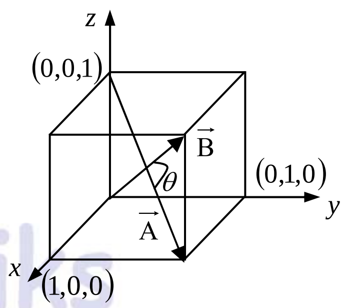
Solution: The face diagonals $\vec{A}$ and $\vec{B}$ are

$\vec{A} = 1\hat{x} + 0\hat{y} + 1\hat{z}; \quad \vec{B} = 0\hat{x} + 1\hat{y} + 1\hat{z}$

So, $\Rightarrow \vec{A}.\vec{B} = 1$

Also, $\Rightarrow \vec{A}.\vec{B} = AB\cos\theta = \sqrt{2}\sqrt{2}\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^0$

Example: Find the angle between the body diagonals of a cube.



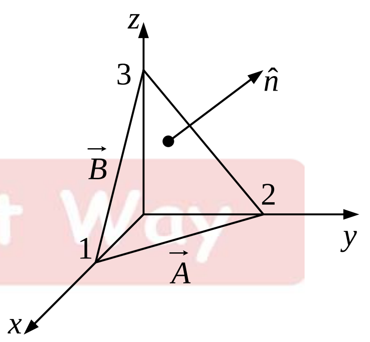
Solution: The body diagonals $\vec{A}$ and $\vec{B}$ are

$\vec{A} = \hat{x} + \hat{y} - \hat{z}; \quad \vec{B} = \hat{x} + \hat{y} + \hat{z}$

So, $\Rightarrow \vec{A}.\vec{B} = 1 + 1 - 1 = 1$

Also, $\Rightarrow \vec{A}.\vec{B} = AB\cos\theta = \sqrt{3}\sqrt{3}\cos\theta \Rightarrow \cos\theta = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right)$

Example: Find the components of the unit vector $\hat{n}$ perpendicular to the plane shown in the figure.



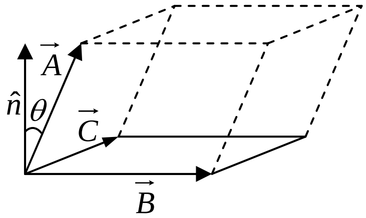
Solution: The vectors $\vec{A}$ and $\vec{B}$ can be defined as

$\vec{A} = -\hat{x} + 2\hat{y}; \quad \vec{B} = -\hat{x} + 3\hat{z} \Rightarrow \hat{n} = \frac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|} = \frac{6\hat{x} + 3\hat{y} + 2\hat{z}}{7}$

1.1.3 Triple Products

Since the cross product of two vectors is itself a vector, it can be dotted or crossed with a third vector to form a triple product.

(i) Scalar triple product: $\vec{A}.\left(\vec{B}\times\vec{C}\right)$



Geometrically $\left|\vec{A}.\left(\vec{B}\times\vec{C}\right)\right|$ is the volume of the parallelepiped generated by $\vec{A}, \vec{B}$ and $\vec{C}$, since $\left|\vec{B}\times\vec{C}\right|$ is the area of the base, and $\left|\vec{A}\cos\theta\right|$ is the altitude. Evidently,

$$\vec{A}.\left(\vec{B}\times\vec{C}\right)=\vec{B}.\left(\vec{C}\times\vec{A}\right)=\vec{C}.\left(\vec{A}\times\vec{B}\right)$$

In component form $\vec{A}.\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

Note that the dot and cross can be interchanged: $\vec{A}.\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}\times\vec{B}\right).\vec{C}$

(ii) Vector triple product: $\vec{A}\times\left(\vec{B}\times\vec{C}\right)$

The vector triple product can be simplified by the so-called **BAC-CAB** rule:

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}.\vec{C}\right)-\vec{C}\left(\vec{A}.\vec{B}\right)$$

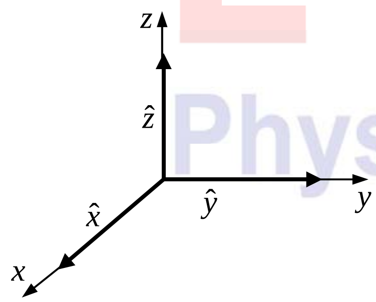
1.2 Coordinate System

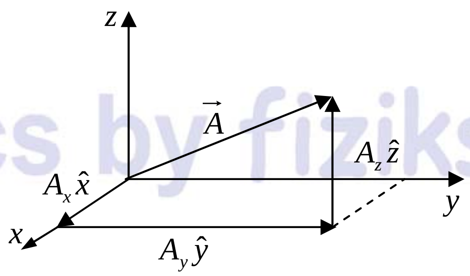
If we want to represent any vector $\vec{A}$ then we need a coordinate system. We have three different coordinate system namely Cartesian coordinate system, spherical polar coordinate system and cylindrical polar coordinate system.

1.2.1 Cartesian Coordinate System

Let $\hat{x}$, $\hat{y}$ and $\hat{z}$ be unit vectors parallel to the x, y and z axis, respectively. An arbitrary vector $\vec{A}$ can be expanded in terms of these basis vectors

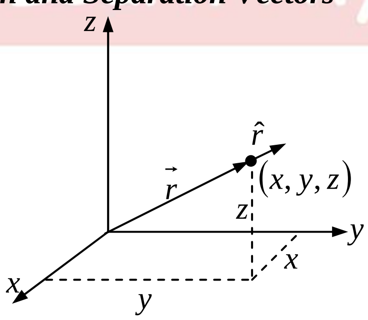
$$\vec{A}=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$$

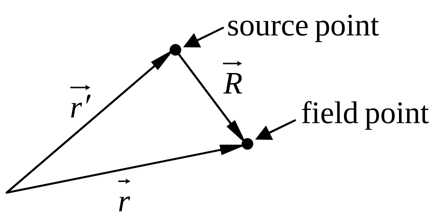




The numbers A_x, A_y, and A_z are called component of $\vec{A}$; geometrically, they are the projections of $\vec{A}$ along the three coordinate axes.

Position and Separation Vectors





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The location of a point in three dimensions can be described by listing its Cartesian coordinates (x, y, z). The vector to that point from the origin is called the position vector:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}.$$

Its magnitude, $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin, and $\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}}$ is a unit vector pointing radially outward.

Note: In electrodynamics one frequently encounters problems involving two points-typically, a **source point ,** $\vec{r}'$, where an electric charge is located, and a **field point,** $\vec{r}$ **,** at which we are calculating the electric or magnetic field. We can define **separation vector** from the source point to the field point by $\vec{R}$;

$$\vec{R} = \vec{r} - \vec{r}'.$$

Its magnitude is $R = |\vec{r} - \vec{r}'|$,

and a unit vector in the direction from $\vec{r}'$ to $\vec{r}$ is $\hat{R} = \frac{\vec{R}}{R} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$.

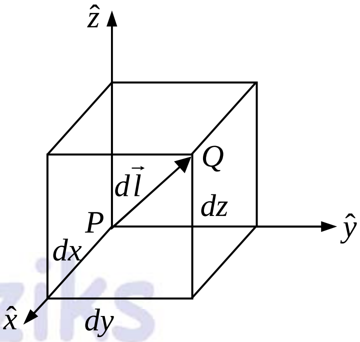
In Cartesian coordinates, $\vec{R} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$

$$|\vec{R}| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \Rightarrow \hat{R} = \frac{(x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

Infinitesimal Displacement Vector $(d\vec{l})$

The infinitesimal displacement vector**,** from point $P(x, y, z)$ to $Q(x + dx, y + dy, z + dz)$, is

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

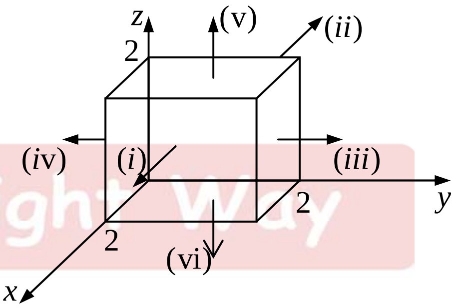


Volume Element $(d\tau)$

Volume element $d\tau = dxdydz$

Area Element $(d\vec{a})$

For closed surface area element is perpendicular to the surface pointing outwards as shown in figure below.



(i) For $x = 2$ plane, $d\vec{a} = dydz\hat{x}$

(ii) For $x = 0$ plane, $d\vec{a} = -dydz\hat{x}$

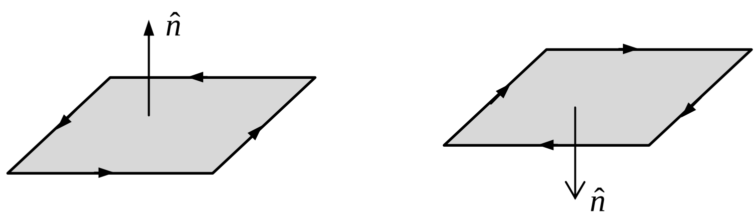
(iii) For $y = 2$ plane, $d\vec{a} = dxdz\hat{y}$

(iv) For $y = 0$ plane, $d\vec{a} = -dxdz\hat{y}$

(v) For $z = 2$ plane, $d\vec{a} = dxdy\hat{z}$

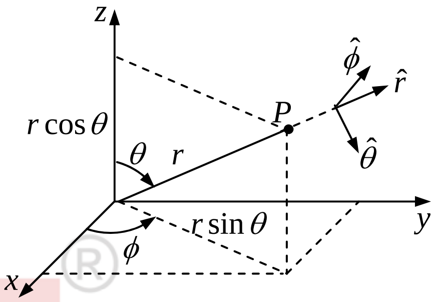
(vi) For $z = 0$ plane, $d\vec{a} = -dxdy\hat{z}$

For open surface area element is shown in figure below (use right hand rule)



1.2.2 Spherical Polar Coordinate System

In spherical polar coordinate any general point P lies on the surface of a sphere. The spherical polar coordinates (r, θ, ϕ) of a point P are defined in figure shown below; r is the distance from the origin (the magnitude of the position vector), θ (the angle drawn from the z axis) is called the polar angle, and ϕ (the angle around from the x axis) is the azimuthal angle.



Their relation to Cartesian coordinates (x, y, z) can be read from the figure:

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

and $r = \sqrt{x^2 + y^2 + z^2}$, $\theta = \cos^{-1}\left(\frac{z}{r}\right)$, $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

The range of r is $0 \to \infty$, θ goes from $0 \to \pi$, and ϕ goes from $0 \to 2\pi$.

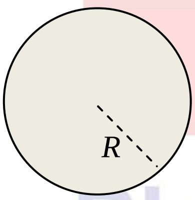
Figure shows three unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$, pointing in the direction of increase of the corresponding coordinates. They constitute an orthogonal (mutually perpendicular) basis set (just like $\hat{x}, \hat{y}, \hat{z}$), and any vector $\vec{A}$ can be expressed in terms of them in the usual way:

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

A_r, A_θ, and A_ϕ are the radial, polar and azimuthal components of $\vec{A}$.

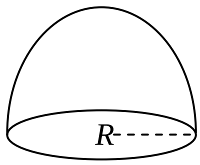
If we have sphere or any part of the sphere, then we can specify r, θ and ϕ. Lets consider some examples shown in figure below:

Sphere



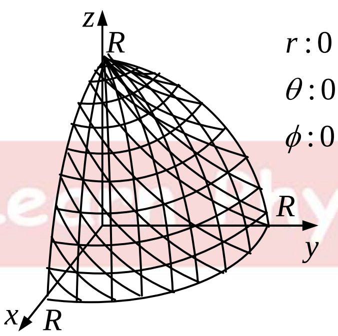
$r : 0 \to R$
$\theta : 0 \to \pi$
$\phi : 0 \to 2\pi$

Hemi-sphere



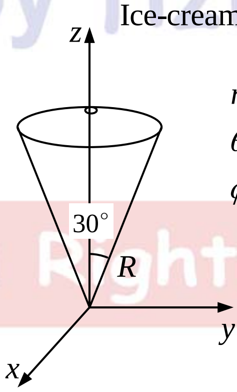
$r : 0 \to R$
$\theta : 0 \to \pi/2$
$\phi : 0 \to 2\pi$

Octant



$r : 0 \to R$
$\theta : 0 \to \pi/2$
$\phi : 0 \to \pi/2$

Ice-creame cone



$r : 0 \to R$
$\theta : 0 \to \pi/6$
$\phi : 0 \to 2\pi$

Infinitesimal Displacement Vector $(d\vec{l})$

An infinitesimal displacement in the $\hat{r}$ direction is simply dr (figure *a*), just as an infinitesimal element of length in the x direction is dx:

$$dl_r = dr$$

On the other hand, an infinitesimal element of length in the $\hat{\theta}$ direction (figure *b*) is $r\,d\theta$

$$dl_\theta = r\,d\theta$$

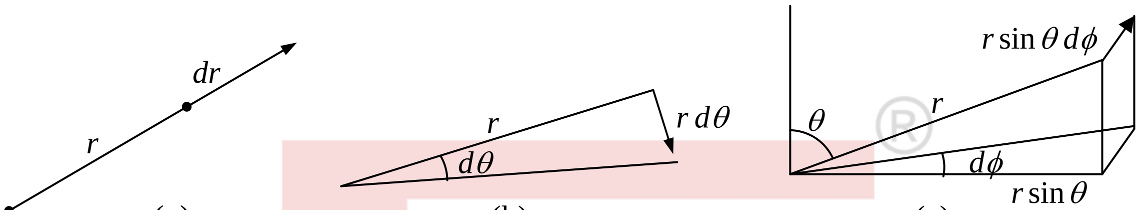
Similarly, an infinitesimal element of length in the $\hat{\phi}$ direction (figure *c*) is $r\sin\theta\, d\phi$

$$dl_\phi = r\sin\theta\, d\phi$$

Thus, the general infinitesimal displacement $d\vec{l}$ is

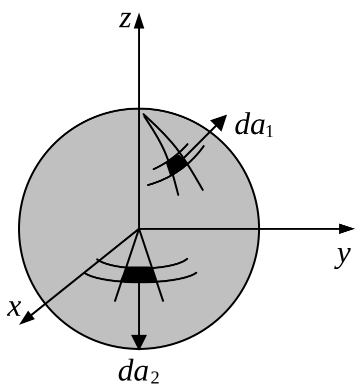
$$d\vec{l} = dr\,\hat{r} + r\,d\theta\hat{\theta} + r\sin\theta\, d\phi\hat{\phi}$$

This plays the role (in line integrals, for example) that $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$ played in Cartesian coordinates.



(a) (b) (c)

Area Element $\left(d\vec{a}\right)$



If we are integrating over the surface of a sphere, for instance, then *r* is constant, whereas θ and ϕ change, so

$$d\vec{a}_1 = dl_\theta dl_\phi \hat{r} = r^2 \sin\theta\, d\theta\, d\phi\,\hat{r}$$

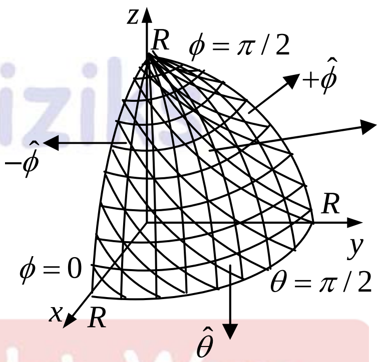
on the other hand, if the surface lies in the *xy* plane, then θ is constant ($\theta = \pi/2$) while *r* and ϕ vary, then

$$d\vec{a}_2 = dl_r dl_\phi \hat{\theta} = r^2\, dr\, d\phi\hat{\theta}$$

If we are integrating over the surface of an octant, for instance, then *r* is constant $\left(r = R\right)$, whereas θ and ϕ change, so,

$d\vec{a}_1 = dl_\theta dl_\phi \hat{r} = R^2 \sin\theta d\theta d\phi\hat{r}$.

Octant



If the surface lies in the *xy* plane, then θ is constant ($\theta = \pi/2$) while r and ϕ vary, then

$$d\vec{a}_2 = dl_r dl_\phi \hat{\theta} = r^2 dr d\phi\hat{\theta}$$

If the surface lies in the *yz* plane, then ϕ is constant ($\phi = \pi/2$) while r and ϕ vary, then

$$d\vec{a}_3 = dl_r dl_\theta \hat{\phi} = r dr d\theta\hat{\phi}$$

If the surface lies in the xz plane, then ϕ is constant ($\phi = 0$) while *r* and ϕ vary, then

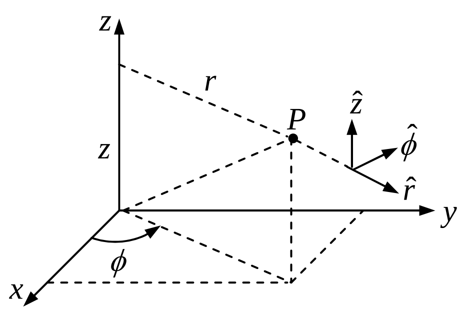
$$d\vec{a}_4 = -dl_r dl_\theta \hat{\phi} = -r dr d\theta\hat{\phi}$$

Volume Element $(d\tau)$

The infinitesimal volume element $d\tau$, in spherical coordinates, is the product of the three infinitesimal displacements:

$$d\tau = dl_r dl_\theta dl_\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

1.2.3 Cylindrical Polar Coordinate System



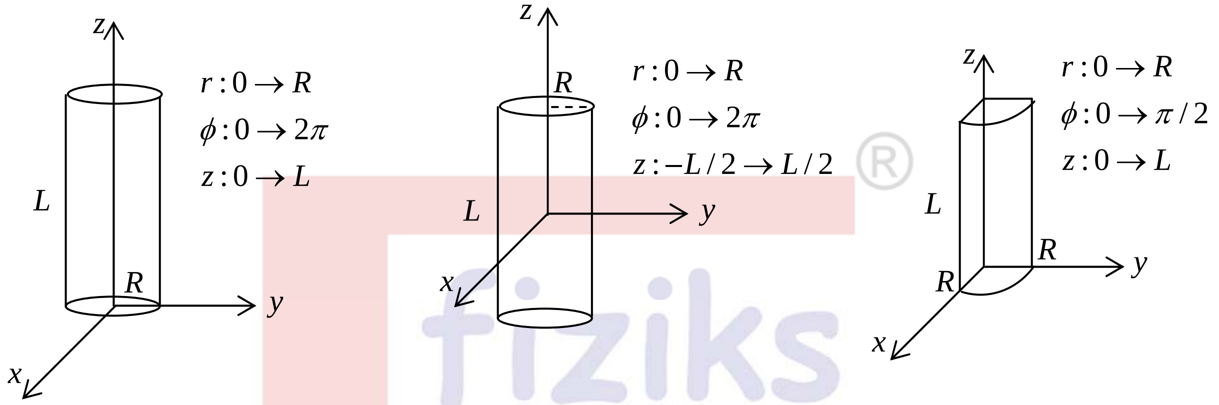
In cylindrical polar coordinate any general point P lies on the surface of a cylinder. The cylindrical coordinates r, ϕ, z of a point *P* are defined in figure. Notice that ϕ has the same meaning as in spherical coordinates, and *z* is the same as Cartesian; *r* is the distance to *P* from the *z* axis, whereas the spherical coordinate *r* is the distance from the origin. The relation to Cartesian coordinates is

$$x = r\cos\phi,\quad y = r\sin\phi\ ,\quad z = z$$

The range of r is $0 \to \infty$, ϕ goes from $0 \to 2\pi$, and z from $-\infty$ to ∞

If we have cylinder or any part of the cylinder, then we can specify r, ϕ and z. Let us consider some examples shown in figure below:

Cylinder Cylinder Quarter Cylinder



Infinitesimal Displacement Vector $\left(d\vec{l}\right)$

If (r,ϕ,z) changes to $(r+dr, \phi+d\phi, z+dz)$,

The infinitesimal displacements are

$$dl_r = dr,\ \ dl_\phi = rd\phi,\ \ dl_z = dz,$$

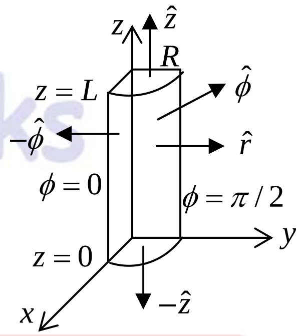
So $$d\vec{l} = dr\,\hat{r} + r\,d\phi\hat{\phi} + dz\,\hat{z}$$

Area Element $\left(d\vec{a}\right)$

If we are integrating over the surface of an quarter cylinder, for instance, then r is constant $(r = R)$, whereas ϕ and z change, so,

$$d\vec{a}_1 = dl_\phi dl_z \hat{r} = rd\phi dz\hat{r}\ .$$

Quarter Cylinder



If the surface lies in the xy plane, then z is constant ($z = L$) while r and ϕ vary, then

$$d\vec{a}_2 = dl_r dl_\phi \hat{z} = rdrd\phi\hat{z}$$

If the surface lies in the xy plane, then z is constant ($z = 0$) while r and ϕ vary, then

$$d\vec{a}_2 = -dl_r dl_\phi \hat{\theta} = -rdrd\phi\hat{z}$$

If the surface lies in the yz plane, then ϕ is constant ($\phi = \pi/2$) while r and z vary, then

$$d\vec{a}_3 = dl_r dl_z \hat{\phi} = drdz\hat{\phi}$$

If the surface lies in the xz plane, then ϕ is constant ($\phi = 0$) while r and z vary, then

$$d\vec{a}_4 = -dl_r dl_z \hat{\phi} = -drdz\hat{\phi}$$

Volume Element $(d\tau)$

The infinitesimal volume element $d\tau$, in cylindrical coordinates, is the product of the three infinitesimal displacements:

$$d\tau = dl_r dl_\phi dl_z = rdrd\phi dz$$

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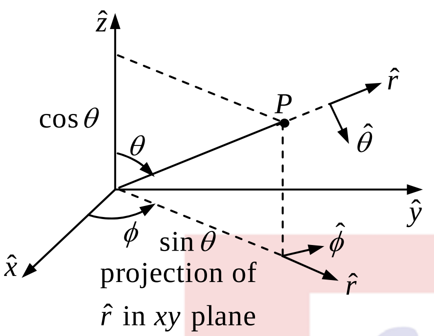
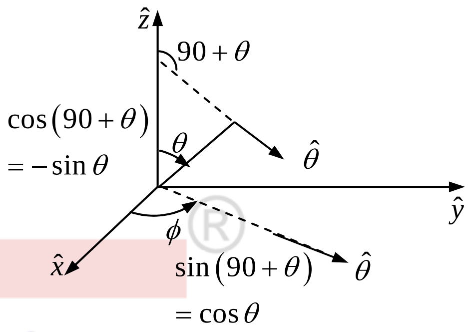
1.2.4 Transformation of a Vector from One System to Other

Transformation of a Vector from Cartesian to Spherical Polar

We can transform any vector $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ in Cartesian coordinates to Spherical polar coordinate as $\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$.





Thus

$$A_r = \vec{A}.\hat{r} = A_x(\hat{x}.\hat{r}) + A_y(\hat{y}.\hat{r}) + A_z(\hat{z}.\hat{r})$$

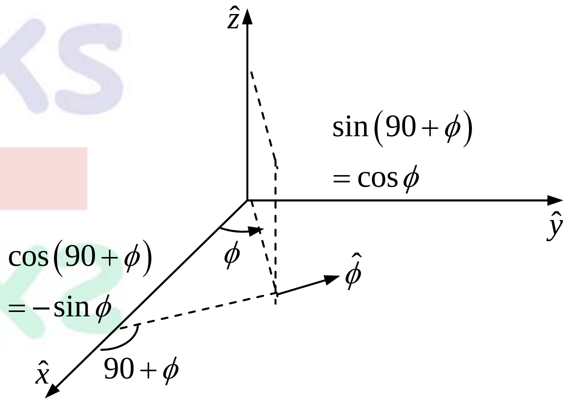
$$A_\theta = \vec{A}.\hat{\theta} = A_x(\hat{x}.\hat{\theta}) + A_y(\hat{y}.\hat{\theta}) + A_z(\hat{z}.\hat{\theta})$$

$$A_\phi = \vec{A}.\hat{\phi} = A_x(\hat{x}.\hat{\phi}) + A_y(\hat{y}.\hat{\phi}) + A_z(\hat{z}.\hat{\phi})$$

where

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

and use table given below:



	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$
$\hat{x}.$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\hat{y}.$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\hat{z}.$	$\cos\theta$	$-\sin\theta$	0

Example: Transform vector $\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$ in Cartesian coordinates to Spherical polar coordinate as $\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$.

Solution:

$$A_r = \vec{A}.\hat{r} = x(\hat{x}.\hat{r}) + y(\hat{y}.\hat{r}) + z(\hat{z}.\hat{r})$$

$$\Rightarrow A_r = r\sin\theta\cos\phi(\sin\theta\cos\phi) + r\sin\theta\sin\phi(\sin\theta\sin\phi) + r\cos\theta(\cos\theta)$$

$$\Rightarrow A_r = r\sin^2\theta(\cos^2\phi + \sin^2\phi) + r\cos^2\theta = r$$

$$A_\theta = \vec{A}.\hat{\theta} = x(\hat{x}.\hat{\theta}) + y(\hat{y}.\hat{\theta}) + z(\hat{z}.\hat{\theta})$$

$$\Rightarrow A_\theta = r\sin\theta\cos\phi(\cos\theta\cos\phi) + r\sin\theta\sin\phi(\cos\theta\sin\phi) + r\cos\theta(-\sin\theta)$$

$$\Rightarrow A_\theta = r\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - r\sin\theta\cos\theta = 0$$

$$A_\phi = \vec{A}.\hat{\phi} = x(\hat{x}.\hat{\phi}) + y(\hat{y}.\hat{\phi}) + z(\hat{z}.\hat{\phi})$$

$$\Rightarrow A_\phi = r\sin\theta\cos\phi(-\sin\phi) + r\sin\theta\sin\phi(\cos\phi) + r\cos\theta\times 0 = 0$$

Thus in spherical polar coordinate $\vec{A} = r\hat{r}$.

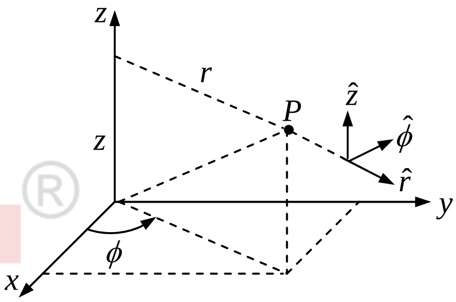
Example: Transform vector $\vec{A} = r\hat{r}$ in Spherical polar coordinate to Cartesian coordinates to as $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$.

Solution: $A_x = \vec{A}.\hat{x} = r(\hat{r}.\hat{x}) = r\sin\theta\cos\phi = x$

$A_y = \vec{A}.\hat{y} = r(\hat{r}.\hat{y}) = r\sin\theta\sin\phi = y$

$A_z = \vec{A}.\hat{z} = r(\hat{r}.\hat{z}) = r\cos\theta = z$

Thus, in Cartesian coordinate $\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$.



Transformation of a Vector from Cartesian to Cylindrical Coordinate

We can transform any vector $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ in Cartesian coordinates to cylindrical coordinates as $\vec{A} = A_r\hat{r} + A_\phi\hat{\phi} + A_z\hat{z}$

Thus,

$A_r = \vec{A}.\hat{r} = A_x(\hat{x}.\hat{r}) + A_y(\hat{y}.\hat{r}) + A_z(\hat{z}.\hat{r})$

$A_\phi = \vec{A}.\hat{\phi} = A_x(\hat{x}.\hat{\phi}) + A_y(\hat{y}.\hat{\phi}) + A_z(\hat{z}.\hat{\phi})$

$A_z = \vec{A}.\hat{z} = A_x(\hat{x}.\hat{z}) + A_y(\hat{y}.\hat{z}) + A_z(\hat{z}.\hat{z})$

where $x = r\cos\phi, \quad y = r\sin\phi, \quad z = z$

use table given below:

	$\hat{r}$	$\hat{\phi}$	$\hat{z}$
$\hat{x}.$	$\cos\phi$	$-\sin\phi$	0
$\hat{y}.$	$\sin\phi$	$\cos\phi$	0
$\hat{z}.$	0	0	1

Example: Transform vector $\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$ in Cartesian coordinates to cylindrical coordinate as $\vec{A} = A_r\hat{r} + A_\phi\hat{\phi} + A_z\hat{z}$.

Solution: $A_r = \vec{A}.\hat{r} = x(\hat{x}.\hat{r}) + y(\hat{y}.\hat{r}) + z(\hat{z}.\hat{r}) \Rightarrow A_r = r\cos\phi(\cos\phi) + r\sin\phi(\sin\phi) + z\times 0 = r$

$A_\phi = \vec{A}.\hat{\phi} = x(\hat{x}.\hat{\phi}) + y(\hat{y}.\hat{\phi}) + z(\hat{z}.\hat{\phi}) \Rightarrow A_\phi = r\cos\phi(-\sin\phi) + r\sin\phi(\cos\phi) + z\times 0 = 0$

$A_z = \vec{A}.\hat{z} = x(\hat{x}.\hat{z}) + y(\hat{y}.\hat{z}) + z(\hat{z}.\hat{z}) \Rightarrow A_z = r\cos\phi\times 0 + r\sin\phi\times 0 + z\times 1 = z$

Thus, in spherical polar coordinate $\vec{A} = r\hat{r} + z\hat{z}$.

Example: Transform vector $\vec{A} = r\hat{r} + z\hat{z}$ in cylindrical coordinate to Cartesian coordinates to as $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$.

Solution: $A_x = \vec{A}.\hat{x} = r(\hat{r}.\hat{x}) + z(\hat{z}.\hat{x}) = r\cos\phi = x$, $A_y = \vec{A}.\hat{y} = r(\hat{r}.\hat{y}) + z(\hat{z}.\hat{y}) = r\sin\phi = y$

$A_z = \vec{A}.\hat{z} = r(\hat{r}.\hat{z}) + z(\hat{z}.\hat{z}) = z$

Thus in Cartesian coordinate $\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$.

1.3 Differential Calculus

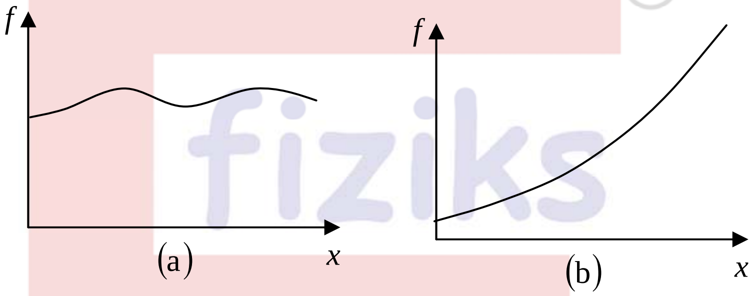
1.3.1 "Ordinary" Derivatives

Suppose we have a function of one variable: $f(x)$ then the derivative, df/dx tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx:

$$df = \left(\frac{df}{dx}\right)dx$$

In words: If we change x by an amount dx, then f changes by an amount df; the derivative is the proportionality factor. For example in figure (a), the function varies slowly with x, and the derivative is correspondingly small. In figure (b), f increases rapidly with x, and the derivative is large, as we move away from $x = 0$.

Geometrical Interpretation: The derivative df/dx is the slope of the graph of f versus x.



1.3.2 Gradient

Suppose that we have a function of three variables-say, $V(x, y, z)$ in a

$$dV = \left(\frac{\partial V}{\partial x}\right)dx + \left(\frac{\partial V}{\partial y}\right)dy + \left(\frac{\partial V}{\partial z}\right)dz.$$

This tells us how V changes when we alter all three variables by the infinitesimal amounts dx, dy, dz. Notice that we do not require an infinite number of derivatives-three will suffice: the partial derivatives along each of the three coordinate directions.

Thus $$dV = \left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)\cdot\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right) = \left(\vec{\nabla}V\right)\cdot\left(d\vec{l}\right),$$

where $\vec{\nabla}V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$ is the gradient of V.

$\vec{\nabla}V$ is a vector quantity, with three components.

Geometrical Interpretation of the Gradient

Like any vector, the gradient has magnitude and direction. To determine its geometrical meaning, let's rewrite

$$dV = \vec{\nabla}V \cdot d\vec{l} = \left|\vec{\nabla}V\right|\left|d\vec{l}\right|\cos\theta$$

where θ is the angle between $\vec{\nabla}V$ and $d\vec{l}$. Now, if we fix the magnitude $\left|d\vec{l}\right|$ and search around in various directions (that is, vary θ), the maximum change in V evidently occurs when $\theta = 0$ (for then $\cos\theta = 1$). That is, for a fixed distance $\left|d\vec{l}\right|$, dV is greatest when one move in the same direction as $\vec{\nabla}V$. Thus:

The gradient $\vec{\nabla}V$ points in the direction of maximum increase of the function V.

Moreover:

The magnitude $\left|\vec{\nabla}V\right|$ gives the slope (rate of increase) along this maximal direction.

Gradient in Spherical polar coordinates $V(r,\theta,\phi)$

$$\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

Gradient in cylindrical coordinates $V(r,\phi,z)$

$$\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}$$

Stationary Points of a Function:

For maxima, minima or saddle point of a function $f(x,y)$: $\vec{\nabla}f = 0$.

So put $f_x = \frac{\partial f}{\partial x} = 0$ and $f_y = \frac{\partial f}{\partial y} = 0$ and solve for $(x,y) = (a,b)$.

Find $f_{xx}(a,b) = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(a,b)}$, $f_{yy}(a,b) = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(a,b)}$ and $f_{xy}(a,b) = \left.\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right|_{(a,b)}$

H-value Test: $H = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$

If $H < 0$, then (a,b) is a saddle point i.e. neither maxima nor minima.

If $H > 0$, then (a,b) is a either maxima or minima.

If $f_{xx}(a,b)$ and $f_{yy}(a,b)$ both are positive then at (a,b) there is minima.

If $f_{xx}(a,b)$ and $f_{yy}(a,b)$ both are negative then at (a,b) there is maxima.

Example: Find stationary points of $f(x,y) = x^4 - 4x^2 + y^2$.

Solution: $f_x = \frac{\partial f}{\partial x} = 4x^3 - 8x = 0 \Rightarrow x = 0, \pm\sqrt{2}$ and $f_y = \frac{\partial f}{\partial y} = 2y = 0 \Rightarrow y = 0$

So points are $(0,0), (\sqrt{2},0), (-\sqrt{2},0)$.

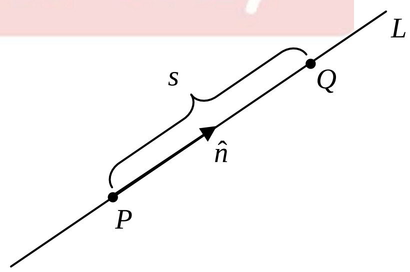
Then $f_{xx} = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 8$, $f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2$, $f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$

	$(0,0)$	$(\sqrt{2},0)$	$(-\sqrt{2},0)$
f_{xx}	-8	16	16
f_{yy}	2	2	2
f_{xy}	0	0	0
H	-16	32	32

$(0,0)$: Saddle point and $(\sqrt{2},0), (-\sqrt{2},0)$ are minima

Directional Derivative: The directional derivative of a function $f(x,y,z)$ at a point P in the direction of a vector $\hat{n}$ is defined by

$$D_{\hat{n}} f = \frac{df}{ds} = \lim_{s\to 0}\frac{f(Q) - f(P)}{s}$$



Here Q is a variable point on the straight-line L in the direction of $\hat{n}$ and s is the distance between P and Q. Also, $s > 0$ if Q lies in the direction of $\hat{n}$, $s < 0$ if Q lies in the direction of $-\hat{n}$ and $s = 0$ if $Q = P$.

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The line L is given by $\vec{r}(s) = x(s)\hat{x} + y(s)\hat{y} + z(s)\hat{z} = \vec{p}_0 + s\hat{n}$ where $\vec{p}_0$ is the position vector of P.

Thus $D_{\hat{n}} f = \frac{df}{ds}$ is the derivative of the function $f(x(s), y(s), z(s))$ with respect to the arc length s of L. Now $D_{\hat{n}} f = \frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z'$

where primes denote derivatives w.r.t. s (which are taken at $s = 0$)

Now $\vec{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z} = \hat{n}$ and $\because \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$

$$\Rightarrow \boxed{D_{\hat{n}} f = \frac{df}{ds} = \hat{n} \cdot (\vec{\nabla} f)_P}$$

Note: If the direction is given by a vector $\vec{a}$ of any length $(\neq 0)$, then

$$\boxed{D_{\vec{a}} f = \frac{df}{ds} = \frac{\vec{a}}{|\vec{a}|} \cdot (\vec{\nabla} f)_P}$$

Example: Find the directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at point $P(2,1,3)$ in the direction of $\vec{a} = \hat{x} - 2\hat{z}$ or $\vec{a} = [1, 0, -2]$.

Solution: $\vec{\nabla} f = 4x\hat{x} + 6y\hat{y} + 2z\hat{z} \Rightarrow (\vec{\nabla} f)_{(2,1,3)} = 8\hat{x} + 6\hat{y} + 6\hat{z}$

$$\Rightarrow D_{\vec{a}} f = \frac{\vec{a}}{|\vec{a}|} \cdot (\vec{\nabla} f)_P = \left(\frac{\hat{x} - 2\hat{z}}{\sqrt{5}}\right) \cdot (8\hat{x} + 6\hat{y} + 6\hat{z}) = \frac{1}{\sqrt{5}}[8 + 0 - 12] = -\frac{4}{\sqrt{5}} = -1.789$$

Thus, minus sign indicates that at P the function f is decreasing in the direction of $\hat{n}$.

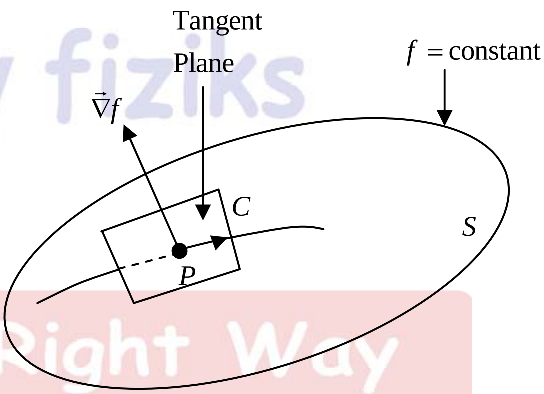
Gradient as a surface Normal Vector:

Let S be a surface given by $f(x, y, z) = C =$ constant (Level surface). Such surface is called a level surface of f and for different C we get different level surfaces. Now let C be a curve on S through a point P of S. As a curve in space, C has representation

$$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$$

For C to lie on the surface S, the components of $\vec{r}(t)$ must satisfy $f(x, y, z) = C$, that is

$$f(x(t), y(t), z(t)) = C$$



Now a tangent vector of C is $\vec{r}'(t) = x'(t)\hat{x} + y'(t)\hat{y} + z'(t)\hat{z}$ and the tangent vectors of all curves on S passing through P will generally form a plane, called tangent plane of S at P. The normal to this plane is called the surface normal to S and P.

Thus $\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0 \Rightarrow \frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' = 0 \Rightarrow \boxed{(\vec{\nabla} f) \cdot \vec{r}' = 0}$

Hence $\vec{\nabla} f$ is orthogonal to all the vectors $\vec{r}'$ in the tangent plane, so that it is a normal vector of S and P.

$$\Rightarrow \hat{n} = \frac{(\vec{\nabla} f)_P}{|\vec{\nabla} f|}$$

Example: Find a unit normal vector $\hat{n}$ of the cone of revolution $z^2 = 4\left(x^2 + y^2\right)$ at the point $P(1,0,2)$.

Solution: The cone is a level surface $f = 0$ of $f(x,y,z) = 4\left(x^2+y^2\right) - z^2$.

Thus $\vec{\nabla} f = 8x\hat{x} + 8y\hat{y} - 2z\hat{z} \Rightarrow \left(\vec{\nabla} f\right)_{(1,0,2)} = 8\hat{x} - 4\hat{z} \Rightarrow \hat{n} = \dfrac{\vec{\nabla} f}{\left|\vec{\nabla} f\right|} = \dfrac{8\hat{x} - 4\hat{z}}{\sqrt{64+16}} = \dfrac{2\hat{x} - \hat{z}}{\sqrt{5}}$

Equation of a Tangent Plane to any Surface:

Let given point be $P(x_0, y_0, z_0)$ and $Q(x,y,z)$ be any point on tangent plane.

Then $\overrightarrow{PQ} = (x - x_0)\hat{x} + (y - y_0)\hat{y} + (z - z_0)\hat{z}$ (tangent vector at P on the surface)

Thus $\left(\vec{\nabla} f\right)$ at P is normal vector at P.

So, $\boxed{\left(\vec{\nabla} f\right)_{\text{at P}} \cdot \overrightarrow{PQ} = 0}$ (Equation of tangent plane at P)

Example: Find the equation of the tangent plane to the surface $x^3 y^2 z = 12$ at the point $P(1,-2,3)$.

Solution: Here given point be $P(1,-2,3)$ and $Q(x,y,z)$ be any point on tangent plane.

$\Rightarrow \overrightarrow{PQ} = (x-1)\hat{x} + (y+2)\hat{y} + (z-3)\hat{z}$

$\vec{\nabla} f = \left(3x^2y^2z\right)\hat{x} + \left(2x^3yz\right)\hat{y} + \left(x^3y^2\right)\hat{z} \Rightarrow \left(\vec{\nabla} f\right)_{(1,-2,3)} = 36\hat{x} - 12\hat{y} + 4\hat{z}$

Thus $\left(\vec{\nabla} f\right)_{(1,-2,3)} \cdot \overrightarrow{PQ} = 0 \Rightarrow 36(x-1) - 12(y+2) + 4(z-3) = 0 \Rightarrow 36x - 12y + 4z = 72$

$\Rightarrow \boxed{9x - 3y + z = 18}$

Example: Find the gradient of a scalar function of position V where $V(x,y,z) = x^2 y + e^z$. Calculate the magnitude of gradient at point $P(1, 5, -2)$.

Solution: $V(x,y,z) = x^2 y + e^z$

$$\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z} = 2xy\hat{x} + x^2\hat{y} + e^z\hat{z}$$

At $P(1, 5, -2) \Rightarrow \vec{\nabla} V = 10\hat{x} + \hat{y} + 0.1353\hat{z} \Rightarrow \left|\vec{\nabla} V\right| = \sqrt{10^2 + 1^2 + 0.1353^2} = 10.056$

Example: Find the unit vector normal to the curve $y = x^2$ at the point (2, 4, 1).

Solution: The equation of curve in the form of surface is given by

$$x^2 - y = 0$$

A constant scalar function V on the surface is given by $V(x, y, z) = x^2 - y$

Taking the gradient

$$\vec{\nabla} V = \vec{\nabla}\left(x^2 - y\right) = \frac{\partial}{\partial x}\left(x^2 - y\right)\hat{x} + \frac{\partial}{\partial y}\left(x^2 - y\right)\hat{y} + \frac{\partial}{\partial z}\left(x^2 - y\right)\hat{z} = 2x\hat{x} - \hat{y}$$

The value of the gradient at point (2, 4, 1), $\vec{\nabla} V = 4\hat{x} - \hat{y}$

The unit vector, as required

$$\hat{n} = \pm\frac{4\hat{x} - \hat{y}}{\left|4\hat{x} - \hat{y}\right|} = \pm\frac{1}{\sqrt{17}}\left(4\hat{x} - \hat{y}\right)$$

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Example: Find the unit vector normal to the surface $xy^3z^2 = 4$ at a point (-1, -1, 2).

Solution: $\vec{\nabla}(xy^3z^2) = \frac{\partial}{\partial x}(xy^3z^2)\hat{x} + \frac{\partial}{\partial y}(xy^3z^2)\hat{y} + \frac{\partial}{\partial z}(xy^3z^2)\hat{z} = (y^3z^2)\hat{x} + (3xy^2z^2)\hat{y} + (2xy^3z)\hat{z}$

At a point (-1, -1, 2), $\vec{\nabla}(xy^3z^2) = -4\hat{x} - 12\hat{y} + 4\hat{z}$

Unit vector normal to the surface

$$\hat{n} = \frac{-4\hat{x} - 12\hat{y} + 4\hat{z}}{\sqrt{(-4)^2 + (-12^2) + 4^2}} = -\frac{1}{\sqrt{11}}(\hat{x} + 3\hat{y} - \hat{z})$$

Example: In electrostatic field problems, the electric field is given by $\vec{E} = -\vec{\nabla}V$, where V is the scalar field potential. If $V = r^2\phi - 2\theta$ in spherical coordinates, then find $\vec{E}$.

Solution: $V = r^2\phi - 2\theta$

In spherical coordinate, $\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$

Substituting the suitable values, $\vec{\nabla}V = 2r\phi\hat{r} - \frac{2}{r}\hat{\theta} + \frac{r^2}{r\sin\theta}\hat{\phi}$

$\Rightarrow \vec{E} = -\vec{\nabla}V = -2r\phi\hat{r} + \frac{2}{r}\hat{\theta} - \frac{r}{\sin\theta}\hat{\phi}$

1.3.3 The Operator $\vec{\nabla}$

The gradient has the formal appearance of a vector, $\vec{\nabla}$, "multiplying" a scalar V:

$$\vec{\nabla}V = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)V$$

The term in parentheses is called "del":

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

We should say that $\vec{\nabla}$ is a vector operator that acts upon V, not a vector that multiplies V.

There are three ways the operator $\vec{\nabla}$ can act:

1. on a scalar function V: $\vec{\nabla}V$ (the **gradient**);
2. on a vector function $\vec{A}$, via the dot product: $\vec{\nabla}\cdot\vec{A}$ (the **divergence**);
3. on a vector function $\vec{A}$, via the cross product: $\vec{\nabla}\times\vec{A}$ (the **curl**).

1.3.4 The Divergence

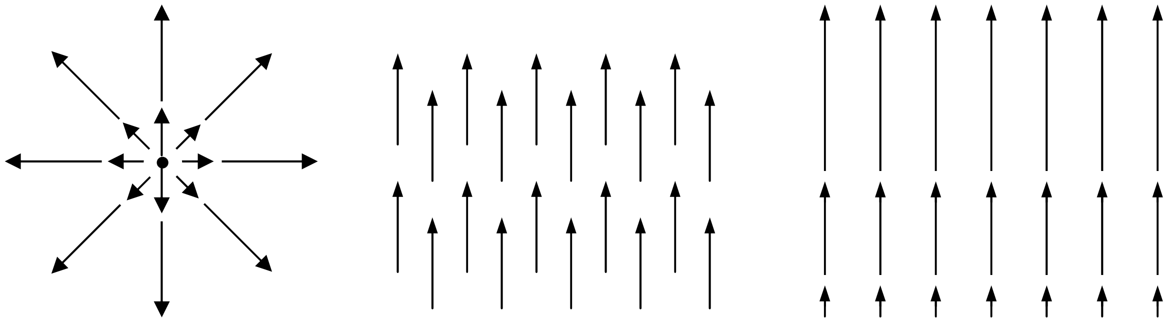
From the definition of $\vec{\nabla}$ we construct the divergence:

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Observe that the divergence of a vector function $\vec{A}$ is itself a scalar $\vec{\nabla}\cdot\vec{A}$. (You can't have the divergence of a scalar: that's meaningless.)

Geometrical Interpretation

$\vec{\nabla}\cdot\vec{A}$ is a measure of how much the vector $\vec{A}$ spreads out (diverges) from the point in question. For example, the vector function in figure (a) has a large (positive) divergence (if the arrows pointed in, it would be a large negative divergence), the function in figure (b) has zero divergence, and the function in figure (c) again has a positive divergence. (Please understand that $\vec{A}$ here is a function-there's a different vector associated with every point in space.)



(a) $\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$ (b) $\vec{A} = \hat{z}$ (c) $\vec{A} = z\hat{z}$

Divergence in Spherical polar coordinates

$$\vec{\nabla}.\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Divergence in cylindrical coordinates

$$\vec{\nabla}.\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

Example: Suppose the function sketched in above figure are $\vec{A} = x\hat{x} + y\hat{y} + z\hat{z}$, $\vec{B} = \hat{z}$ and $\vec{C} = z\hat{z}$. Calculate their divergences.

Solution:

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

$$\vec{\nabla}\cdot\vec{B} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0+0+0 = 0$$

$$\vec{\nabla}\cdot\vec{C} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 0+0+1 = 1$$

Example: Given

(i) $\vec{A} = 2xy\hat{x} + z\hat{y} + yz^2\hat{z}$, find $\vec{\nabla}\cdot\vec{A}$ at $(2,-1,3)$

(ii) $\vec{A} = 2r\cos^2\phi\hat{r} + 3r^2\sin z\hat{\phi} + 4z\sin^2\phi\hat{z}$, find $\vec{\nabla}\cdot\vec{A}$

(iii) $\vec{A} = 10\hat{r} + 5\sin\theta\hat{\theta}$, Find $\vec{\nabla}\cdot\vec{A}$

Solution: (i) In Cartesian coordinates $\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

$A_x = 2xy, A_y = z, A_z = yz^2 \Rightarrow \vec{\nabla}\cdot\vec{A} = 2y + 0 + 2yz$, At $(2,-1,3)$, $\vec{\nabla}\cdot\vec{A} = -2-6 = -8$

(ii) In cylindrical coordinates $\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$

$$A_r = 2r\cos^2\phi, A_\phi = 3r^2\sin z, A_z = 4z\sin^2\phi$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r}4r\cos^2\phi + 0 + 4\sin^2\phi = 4\left(\cos^2\phi + \sin^2\phi\right) = 4$$

(iii) In spherical coordinates, $\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$$A_r = 10, A_\theta = 5\sin\theta, A_\phi = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}20r + \frac{1}{r\sin\theta}10\sin\theta\cos\theta = \left(2+\cos\theta\right)\left(10/r\right)$$

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1.3.5 The Curl

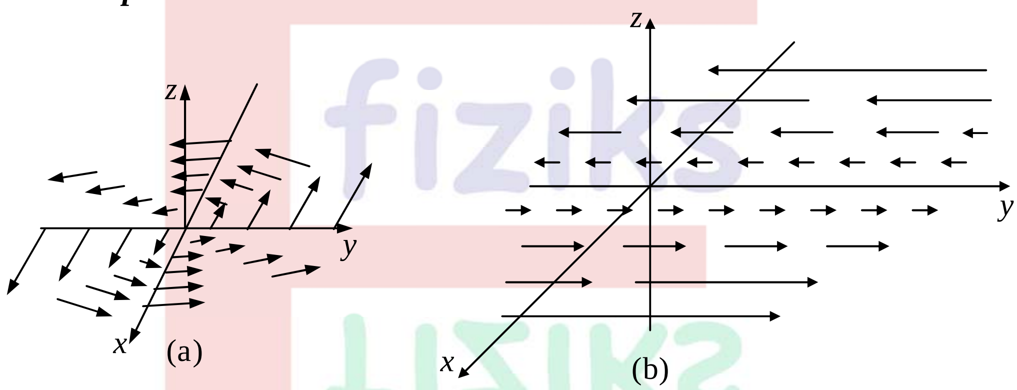
From the definition of $\vec{\nabla}$ we construct the curl

$$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ A_x & A_y & A_z\end{vmatrix}$$

$$=\hat{x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\hat{y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\hat{z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

Notice that the curl of a vector function $\vec{A}$ is, like any cross product, a vector. (You cannot have the curl of a scalar; that's meaningless.)

Geometrical Interpretation



$\vec{\nabla}\times\vec{A}$ is a measure of how much the vector $\vec{A}$ "curls around" the point in question. Figure shown below have a substantial curl, pointing in the z-direction, as the natural right-hand rule would suggest.

Curl in Spherical polar coordinates $\vec{\nabla}\times\vec{A}=\dfrac{1}{r^2\sin\theta}\begin{pmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{pmatrix}$

Curl in cylindrical coordinates $\vec{\nabla}\times\vec{A}=\dfrac{1}{r}\begin{pmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z}\\ A_r & rA_\phi & A_z\end{pmatrix}$

Example: Suppose the function sketched in above figure are $\vec{A}=-y\hat{x}+x\hat{y}$ and $\vec{B}=x\hat{y}$. Calculate their curls.

Solution: $\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ -y & x & 0\end{vmatrix}=2\hat{z}$ and $\vec{\nabla}\times\vec{B}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 0 & x & 0\end{vmatrix}=\hat{z}$

As expected, these curls point in the +z direction. (Incidentally, they both have zero divergence, as you might guess from the pictures: nothing is "spreading out"…. it just "curls around.")

Example: Given a vector function $\vec{A}=(x+c_1z)\hat{x}+(c_2x-3z)\hat{y}+(x+c_3y+c_4z)\hat{z}$.

(a) Calculate the value of constants c_1, c_2, c_3 if $\vec{A}$ is irrotational.

(b) Determine the constant c_4 if $\vec{A}$ is also solenoidal.

(c) Determine the scalar potential function V, whose negative gradient equals $\vec{A}$.

Solution: If $\vec{A}$ is irrotational then, $\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (x+c_1 z) & (c_2 x-3z) & (x+c_3 y+c_4 z)\end{vmatrix}=0$

$\Rightarrow \vec{\nabla}\times\vec{A}=(c_3+3)\hat{x}-(1-c_1)\hat{y}+(c_2-0)\hat{z}=0 \Rightarrow c_1=1, c_2=0, c_3=-3$

(b) If $\vec{A}$ is solenoidal, $\vec{\nabla}\cdot\vec{A}=0\Rightarrow \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}=1+0+c_4=0\Rightarrow c_4=-1$

(c) $\vec{A}=-\vec{\nabla}V=-\frac{\partial V}{\partial x}\hat{x}-\frac{\partial V}{\partial y}\hat{y}-\frac{\partial V}{\partial z}\hat{z}$

$\vec{A}=(x+z)\hat{x}+(-3z)\hat{y}+(x-3y-z)\hat{z}\Rightarrow \frac{\partial V}{\partial x}=-x-z\Rightarrow \text{V}=-\frac{x^2}{2}-xz+f_1(y,z),$

$\frac{\partial V}{\partial y}=3z\Rightarrow \text{V}=3yz+f_2(x,z),\ \frac{\partial V}{\partial z}=-x+3y+z\Rightarrow V=-xz+3yz+\frac{z^2}{2}+f_3(x,y)$

Examination of above expressions of V gives a general value of

$$V=-\frac{x^2}{2}-xz+3yz+\frac{z^2}{2}$$

Example: Find the curl of the vector $\vec{A}=\left(e^{-r}/r\right)\hat{\theta}$

Solution: $\vec{A}=\left(e^{-r}/r\right)\hat{\theta}\Rightarrow A_r=0,\ A_\theta=\left(e^{-r}/r\right),\ A_\phi=0$

$$\vec{\nabla}\times\vec{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}=-\frac{e^{-r}}{r}\hat{\phi}$$

Example: Find the nature of the following fields by determining divergence and curl.

(i) $\vec{F}_1=30\hat{x}+2xy\hat{y}+5xz^2\hat{z}$

(ii) $\vec{F}_2=\left(\frac{150}{r^2}\right)\hat{r}+10\hat{\phi}$ (Cylindrical coordinates)

Solution:

(i) $\vec{F}_1=30\hat{x}+2xy\hat{y}+5xz^2\hat{z}\Rightarrow\vec{\nabla}\cdot\vec{F}_1=\frac{\partial F_{1x}}{\partial x}+\frac{\partial F_{1y}}{\partial y}+\frac{\partial F_{1z}}{\partial z}=2x(1+5z)$

Divergence exists, so the field is non-solenoidal.

$\vec{\nabla}\times\vec{F}_1=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 30 & 2xy & 5xz^2\end{vmatrix}=-5z^2\hat{y}+2y\hat{z}$. The field has a curl so it is rotational.

(ii) $\vec{F}_2=\left(\frac{150}{r^2}\right)\hat{r}+10\hat{\phi}$ in cylindrical coordinates.

In cylindrical coordinates, Divergence $\vec{\nabla}\cdot\vec{F}_2=\frac{1}{r}\frac{\partial}{\partial r}\left(rF_{2r}\right)+\frac{1}{r}\frac{\partial F_{2\phi}}{\partial\phi}+\frac{\partial F_{2z}}{\partial z}=\frac{-150}{r^3}$

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The field is non-solenoid.

$$\vec{\nabla}\times\vec{F}_2=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z}\\ \left(\frac{150}{r^2}\right) & 10r & 0\end{vmatrix}=\frac{10}{r}\hat{z}.$$ $\vec{F}_2$ has non-zero curl so it is rotational.

1.3.6 Product Rules

The calculation of ordinary derivatives is facilitated by a number of general rules, such as the

sum rule: $\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx}$,

the rule for multiplying by a constant: $\frac{d}{dx}(kf)=k\frac{df}{dx}$,

the product rule: $\frac{d}{dx}(fg)=f\frac{dg}{dx}+g\frac{df}{dx}$,

and the quotient rule: $\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{g\frac{df}{dx}-f\frac{dg}{dx}}{g^2}$.

Similar relations hold for the vector derivatives. Thus,

$$\vec{\nabla}(f+g)=\vec{\nabla}f+\vec{\nabla}g,\quad \vec{\nabla}\cdot\left(\vec{A}+\vec{B}\right)=\left(\vec{\nabla}\cdot\vec{A}\right)+\left(\vec{\nabla}\cdot\vec{B}\right),$$

$$\vec{\nabla}\times\left(\vec{A}+\vec{B}\right)=\left(\vec{\nabla}\times\vec{A}\right)+\left(\vec{\nabla}\times\vec{B}\right),$$

and $$\vec{\nabla}(kf)=k\vec{\nabla}f,\quad \vec{\nabla}\cdot\left(k\vec{A}\right)=k\left(\vec{\nabla}\cdot\vec{A}\right),\qquad \vec{\nabla}\times\left(k\vec{A}\right)=k\left(\vec{\nabla}\times\vec{A}\right),$$

as you can check for yourself. The product rules are not quite so simple. There are two ways to construct a scalar as the product of two functions:

fg (product of two scalar functions),

$\vec{A}\cdot\vec{B}$ (Dot product of two vectors),

and two ways to make a vector:

$f\vec{A}$ (Scalar time's vector),

$\vec{A}\times\vec{B}$ (Cross product of two vectors),

Accordingly, there are six product rules,

Two for gradients

(i) $\vec{\nabla}(fg)=f\vec{\nabla}g+g\vec{\nabla}f$, (ii) $\vec{\nabla}\left(\vec{A}\cdot\vec{B}\right)=\vec{A}\times\left(\vec{\nabla}\times\vec{B}\right)+\vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)+\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}+\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A}$,

Two for divergences

(iii) $\vec{\nabla}\cdot\left(f\vec{A}\right)=f\left(\vec{\nabla}\cdot\vec{A}\right)+\vec{A}\cdot\left(\vec{\nabla}f\right)$, (iv) $\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)$,

And two for curls

(v) $\vec{\nabla}\times\left(f\vec{A}\right)=f\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\times\left(\vec{\nabla}f\right)$, (vi) $\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A}-\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}+\vec{A}\left(\vec{\nabla}\cdot\vec{B}\right)-\vec{B}\left(\vec{\nabla}\cdot\vec{A}\right)$,

It is also possible to formulate three quotient rules:

$$\vec{\nabla}\left(\frac{f}{g}\right)=\frac{g\vec{\nabla}f-f\vec{\nabla}g}{g^2},\quad \vec{\nabla}\cdot\left(\frac{\vec{A}}{g}\right)=\frac{g\left(\vec{\nabla}\cdot\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}g\right)}{g^2},\quad \vec{\nabla}\times\left(\frac{\vec{A}}{g}\right)=\frac{g\left(\vec{\nabla}\times\vec{A}\right)+\vec{A}\times\left(\vec{\nabla}g\right)}{g^2}.$$

1.3.7 Second Derivatives

The gradient, the divergence, and the curl are the only first derivatives we can make with $\vec{\nabla}$; by applying $\vec{\nabla}$ twice we can construct five species of second derivatives. The gradient $\vec{\nabla}V$ is a vector, so we can take the *divergence* and *curl* of it:

(1) Divergence of gradient: $\vec{\nabla}\cdot\left(\vec{\nabla}V\right)$

$$\vec{\nabla}\cdot\left(\vec{\nabla}V\right)=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial V}{\partial x}\hat{x}+\frac{\partial V}{\partial y}\hat{y}+\frac{\partial V}{\partial z}\hat{z}\right)=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}.$$

This object, which we write $\nabla^2 V$ for short, is called the **Laplacian** of *V*. Notice that the Laplacian of a *scalar V* is a *scalar*.

Laplacian in Spherical polar coordinates

$$\nabla^2 V=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial\phi^2}\right)$$

Laplacian in cylindrical coordinates

$$\nabla^2 V=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}$$

Occasionally, we shall speak of the Laplacian of a *vector*, $\nabla^2\vec{A}$. By this we mean a *vector* quantity whose *x*-component is the Laplacian of A_x, and so on:

$$\nabla^2\vec{A}\equiv\left(\nabla^2 A_x\right)\hat{x}+\left(\nabla^2 A_y\right)\hat{y}+\left(\nabla^2 A_z\right)\hat{z}.$$

This is nothing more than a convenient extension of the meaning of ∇^2.

(2) Curl of gradient: $\vec{\nabla}\times\left(\vec{\nabla}V\right)$

The divergence $\vec{\nabla}\cdot\vec{A}$ is a *scalar*-all we can do is taking its gradient.

The curl of a gradient is always *zero*: $\vec{\nabla}\times\left(\vec{\nabla}V\right)=0$.

(3) Gradient of divergence: $\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)$

The curl $\vec{\nabla}\times\vec{A}$ is a *vector*, so we can take its *divergence* and *curl*.

Notice that $\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)$ is not the same as the Laplacian of a vector:

$$\nabla^2\vec{A}=\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{A}\neq\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right).$$

(4) Divergence of curl: $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)$

The divergence of a curl, like the curl of a gradient, is *always zero*:

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right)=0.$$

(5) Curl of curl: $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)$

As you can check from the definition of $\vec{\nabla}$:

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)-\nabla^2\vec{A}.$$

So curl-of-curl gives nothing new; the first term is just number (3) and the second is the Laplacian (of a vector).

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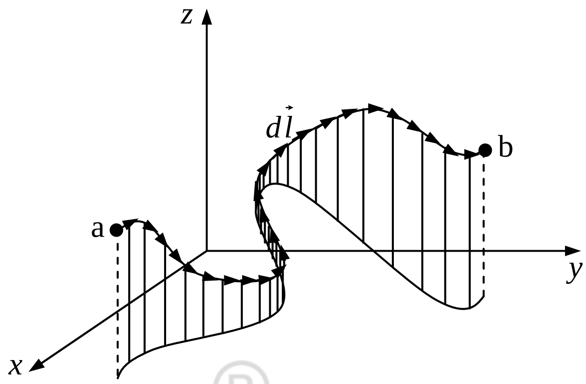
1.4 Integral Calculus

1.4.1 Line, Surface, and Volume Integrals

(a) Line Integrals



A line integral is an expression of the form

$$\int_a^b \vec{A}\cdot d\vec{l},$$

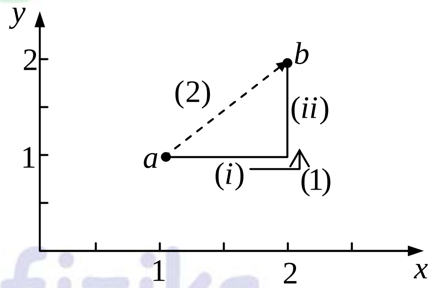
where $\vec{A}$ is a vector function, $d\vec{l}$ is the infinitesimal displacement vector and the integral is to be carried out along a prescribed path P from point a to point b. If the path in question forms a closed loop (that is, if $b = a$), put a circle on the integral sign:

$$\oint \vec{A}\cdot d\vec{l}.$$

At each point on the path we take the dot product of $\vec{A}$ (evaluated at that point) with the displacement $d\vec{l}$ to the next point on the path. The most familiar example of a line integral is the work done by a force $\vec{F}$:

$$W = \int \vec{F}\cdot d\vec{l}$$

Ordinarily, the value of a line integral depends critically on the particular path taken from a to b, but there is an important special class of vector functions for which the line integral is independent of the path, and is determined entirely by the end points (A *force* that has this property is called ***conservative***.)



Example: Calculate the line integral of the function $\vec{A} = y^2\hat{x} + 2x(y+1)\hat{y}$ from the point a = (1, 1, 0) to the point b = (2, 2, 0), along the paths (1) and (2) as shown in figure. What is $\oint \vec{A}\cdot d\vec{l}$ for the loop that goes from a to b along (1) and returns to a along (2)?

Solution: Since $d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$. Path (1) consists of two parts. Along the "horizontal" segment $dy = dz = 0$, so

(i) $d\vec{l} = dx\hat{x},\quad y = 1,\quad \vec{A}\cdot d\vec{l} = y^2 dx = dx,$ so $\int \vec{A}\cdot d\vec{l} = \int_1^2 dx = 1$

On the "vertical" stretch $dx = dz = 0$, so

(ii) $d\vec{l} = dy\hat{y},\quad x = 2,\quad \vec{A}\cdot d\vec{l} = 2x(y+1)dy = 4(y+1)dy,$ so $\int \vec{A}\cdot d\vec{l} = 4\int_1^2 (y+1)dy = 10$.

By path (1), then,

$$\int_a^b \vec{A}\cdot d\vec{l} = 1 + 10 = 11$$

Meanwhile, on path (2) $x = y$, $dx = dy$, and $dz = 0$, so

$$d\vec{l} = dx\hat{x} + dx\hat{y},\quad \vec{A}\cdot d\vec{l} = x^2dx + 2x(x+1)dx = (3x^2 + 2x)dx$$

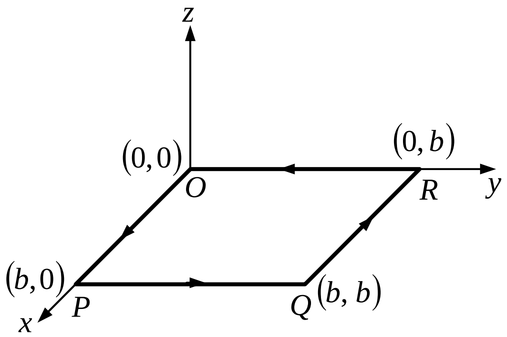
so

$$\int_a^b \vec{A}\cdot d\vec{l} = \int_1^2 (3x^2 + 2x)dx = \left(x^3 + x^2\right)\Big|_1^2 = 10$$

For the loop that goes out (1) and back (2), then,

$$\oint \vec{A}\cdot d\vec{l} = 11 - 10 = 1$$

Example: Find the line integral of the vector $\vec{A} = (x^2 - y^2)\hat{x} + 2xy\hat{y}$ around a square of side 'b' which has a corner at the origin, one side on the x axis and the other side on the y axis.



Solution: In a Cartesian coordinate system $d\hat{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$, $\vec{A} = (x^2 - y^2)\hat{x} + 2xy\hat{y}$

$$\oint_{OPQRO} \vec{A} \cdot d\vec{l} = \oint_{OPQRO} \left[(x^2 - y^2)dx + 2xydy\right]$$

Along OP, $y = 0, dy = 0 \Rightarrow \int_{OP} \vec{A} \cdot d\vec{l} = \int_{x=0}^{b} x^2 dx = \frac{b^3}{3}$

Along PQ, $x = b,\ dx = 0 \Rightarrow \int_{PQ} \vec{A} \cdot d\vec{l} = \int_{y=0}^{b} 2b\, y\, dy = b^3$

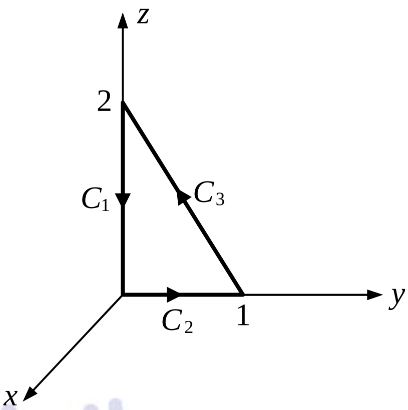
Along QR, $y = b,\ dy = 0 \Rightarrow \int_{QR} \vec{A} \cdot d\vec{l} = \int_{x=b}^{0} (x^2 - b^2)dx = \left(\frac{x^2}{3} - b^2 x\right)\Bigg|_{x=b}^{0} = \frac{2}{3}b^3$

Along RO, $x = 0,\ dx = 0 \Rightarrow \int_{RO} \vec{A} \cdot d\vec{l} = 0$

$$\oint \vec{A} \cdot d\vec{l} = \int_{OP} \vec{A} \cdot d\vec{l} + \int_{PQ} \vec{A} \cdot d\vec{l} + \int_{QR} \vec{A} \cdot d\vec{l} + \int_{RO} \vec{A} \cdot d\vec{l}$$

$$= \frac{b^3}{3} + b^2 + \frac{2}{3}b^3 + 0 = 2b^3$$

Example: Compute the line integral $\vec{F} = 6\hat{x} + yz^2\hat{y} + (3y + z)\hat{z}$ along the triangular path shown in figure.



Solution: Line Integral $\oint \vec{F}.d\vec{l} = \int_{C_1} \vec{F}.d\vec{l} + \int_{C_2} \vec{F}.d\vec{l} + \int_{C_3} \vec{F}.d\vec{l}$

On path C_1, $x = 0,\ y = 0,\ d\vec{l} = dz\hat{z}$

$$\int_{C_1} \vec{F} \cdot d\vec{l} = \int_{z=2}^{0} \left[6\hat{x} + yz^2\hat{y} + (3y + z)\hat{z}\right] \cdot dz\hat{z} = \int_{z=2}^{0} zdz = \frac{z^2}{2}\Bigg|_2^0 = -\frac{4}{2} = -2$$

On path C_2, $x = 0,\ z = 0,\ d\vec{l} = dy\hat{y} \Rightarrow \int_{C_2} \vec{F} \cdot d\vec{l} = \int_{y=0}^{1} yz^2 dy = 0$

On path C_3 the slope of line is-2 and intercept on z axis is $2 \Rightarrow z = -2y + 2 = 2(1 - y)$ and the connecting points are (0, 1, 0) and (0, 0, 2)

On C_3, x=0, $dx = 0$ $\quad d\vec{l} = dy\hat{y} + dz\hat{z}$

$$\int_{C_3} \vec{F} \cdot d\vec{l} = \int_{C_3} (yz^2)dy + (3y + z)dz = \int_{y=1}^{0} y[2(1 - y)]^2 dy + \int_{z=0}^{2} \left[3\left(\frac{2 - z}{2}\right) + z\right]dz$$

$$= \int_1^0 (4y + 4y^3 - 8y^2)dy + \int_{z=0}^{2} \left(3 - \frac{z}{2}\right)dz = 4\frac{y^2}{2}\Bigg|_1^0 + \frac{4y^4}{4}\Bigg|_1^0 - \frac{8y^3}{3}\Bigg|_1^0 + 3z\Big|_0^2 - \frac{1}{2}\frac{z^2}{2}\Bigg|_0^2$$

$$= -2 - 1 + \frac{8}{3} + 6 - 1 = \frac{14}{3} \Rightarrow \oint \vec{F}.d\vec{l} = \int_{C_1} \vec{F}.d\vec{l} + \int_{C_2} \vec{F}.d\vec{l} + \int_{C_3} \vec{F}.d\vec{l} = -2 + 0 + \frac{14}{3} = \frac{8}{3}$$

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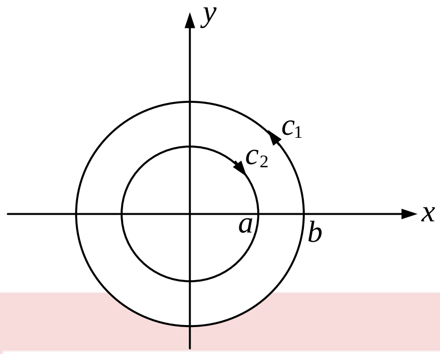
Example: Given $\vec{A} = 2r\cos\phi\,\hat{r} + r\hat{\phi}$ in cylindrical coordinates. Find $\oint_{C_1}\vec{A}\cdot d\vec{l} + \oint_{C_2}\vec{A}\cdot d\vec{l}$ where c_1 and c_2 are contours shown in figure.



Solution: In cylindrical coordinate system $d\vec{l} = dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z}$, $\vec{A} = 2r\cos\phi\,\hat{r} + r\hat{\phi}$

$$\vec{A}\cdot d\vec{l} = 2r\cos\phi dr + r^2 d\phi$$

In figure on curve c_1, ϕ varies from 0 to $2\pi, r = b$ and $dr = 0$

$$\oint_{c_1}\vec{A}\cdot d\vec{l} = \int_{\substack{\phi=0\\ r=b}}^{2\pi} r^2 d\phi = 2\pi b^2$$

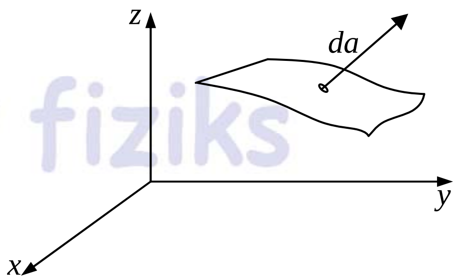
On curve $c_2, r = a, \phi$ varies from 0 to -2π, and $dr = 0 \Rightarrow \oint_{c_2}\vec{A}\cdot d\vec{l} = \int_{\substack{\phi=0\\ r=a}}^{-2\pi} r^2 d\phi = -2\pi a^2$

So, $$\oint_{c_1}\vec{A}\cdot d\vec{l} + \oint_{c_2}\vec{A}\cdot d\vec{l} = 2\pi\left(b^2 - a^2\right)$$

(b) Surface Integrals

A surface integral is an expression of the form

$$\int_S \vec{A}\cdot d\vec{a}$$

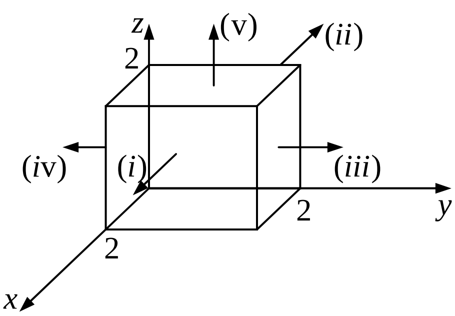


where $\vec{A}$ is again some vector function, and $d\vec{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface(as shown in figure). There are, of course, two directions perpendicular to any surface, so the sign of a surface integral is intrinsically ambiguous. If the surface is closed then "outward" is positive, but for open surfaces it's arbitrary.

If $\vec{A}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \vec{A}\cdot d\vec{a}$ represents the total mass per unit time passing through the surface-hence the alternative name, "flux."

Ordinarily, the value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is independent of the surface, and is determined entirely by the boundary line.

Example: Calculate the surface integral of $\vec{A} = 2xz\hat{x} + (x+2)\hat{y} + y(z^2 - 3)\hat{z}$ over five sides (excluding the bottom) of the cubical box (side 2) as shown in figure. Let "upward and outward" be the positive direction, as indicated by the arrows.



Solution: Taking the sides one at a time:

(i) $x = 2$, $d\vec{a} = dydz\hat{x}$, $\vec{A}\cdot d\vec{a} = 2xzdydz = 4zdydz$, so

$\int \vec{A}\cdot d\vec{a} = 4\int_0^2 dy \int_0^2 zdz = 16$.

(ii) $x=0,\ d\vec{a} = -dydz\hat{x},\ \vec{A}\cdot d\vec{a} = -2xzdydz = 0,$ so $\int \vec{A}\cdot d\vec{a} = 0$.

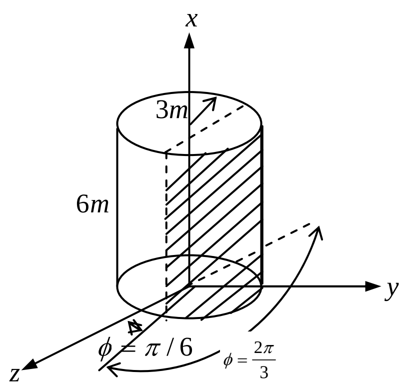
(iii) $y=2,\ d\vec{a} = dx\,dz\,\hat{y},\ \vec{A}\cdot d\vec{a} = (x+2)dx\,dz,$ so $\int \vec{A}\cdot d\vec{a} = \int_0^2 (x+2)dx \int_0^2 dz = 12$.

(iv) $y=0,\ d\vec{a} = -dx\,dz\,\hat{y},\ \vec{A}\cdot d\vec{a} = -(x+2)dx\,dz,$ so $\int \vec{A}\cdot d\vec{a} = -\int_0^2 (x+2)dx \int_0^2 dz = -12$.

(v) $z=2,\ d\vec{a} = dx\,dy\,\hat{z},\ \vec{A}\cdot d\vec{a} = y(z^2-3)dx\,dy = y\,dx\,dy,$ so $\int \vec{A}\cdot d\vec{a} = \int_0^2 dx \int_0^2 ydy = 4$

Evidently the total flux is $\int_{\text{surface}} \vec{A}\cdot d\vec{a} = 16+0+12-12+4 = 20$

Example: Use the cylindrical coordinate system to find the area of a curved surface on the right circular cylinder having radius = 3 m and height = 6 m and $30^0 \le \phi \le 120^0$.



Solution: From figure, surface area is required for a cylinder when r = 3m, z = 0 to 6m,

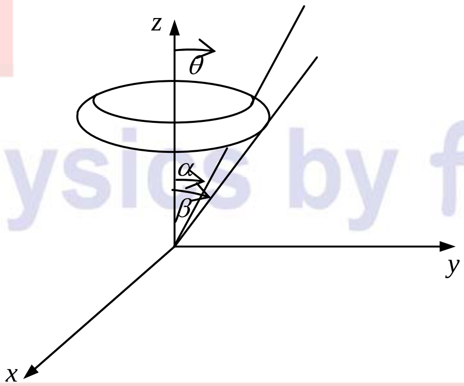
$$30^0 \le \phi \le 120^0 \text{ or } \frac{\pi}{6} \le \phi \le \frac{2\pi}{3}$$

In cylindrical coordinate system, the elemental surface area as scalar is $d\vec{a} = r\,d\phi\,dz\hat{r}$

Taking the magnitude only

$$A = \int_S da = \int_{\phi=\pi/6}^{2\pi/3} \int_{z=0}^{6} r\,d\phi\,dz = 3\left(\frac{2\pi}{3} - \frac{\pi}{6}\right)6 = 9\pi\,m^2$$

Example: Use spherical coordinate system to find the area of the strip $\alpha \le \theta \le \beta$ on the spherical shell of radius 'a'. Calculate the area when $\alpha = 0$ and $\beta = \pi$.



Solution: Sphere has radius 'a' and θ varies between α and β.

For fixed radius the elemental surface is $da = (r\sin\theta\,d\phi)(r\,d\theta) = r^2 \sin\theta\,d\theta\,d\phi$

Area $A = \int_{\theta=\alpha}^{\beta} \int_{\phi=0}^{2\pi} r^2 \sin\theta\,d\theta\,d\phi = 2\pi a^2 \int_{\theta=\alpha}^{\beta} \sin\theta\,d\theta = 2\pi a^2 (\cos\alpha - \cos\beta)$

For $\alpha = 0, \beta = \pi$, Area $= 2\pi a^2(1+1) = 4\pi a^2$, is surface area of the sphere.

(c) Volume Integrals

A volume integral is an expression of the form

$$\int_V T\, d\tau,$$

where T is a scalar function and $d\tau$ is an infinitesimal volume element. In Cartesian coordinates,

$$d\tau = dx\,dy\,dz.$$

For example, if T is the density of a substance (which might vary from point to point) then the volume integral would give the total mass. Occasionally we shall encounter volume integrals of vector functions:

$$\int \vec{A}\, d\tau = \int \left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right) d\tau = \hat{x}\int A_x\, d\tau + \hat{y}\int A_y\, d\tau + \hat{z}\int A_z\, d\tau\,;$$

because the unit vectors are constants, they come outside the integral.

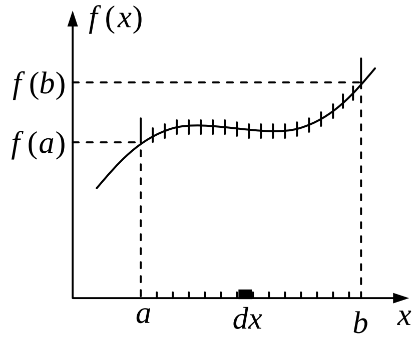
1.4.2 The Fundamental Theorem of Calculus

Suppose $f(x)$ is a function of one variable. The **fundamental theorem of calculus** states:

$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a) \text{ or}$$

$$\int_a^b F(x)\, dx = f(b) - f(a)$$

where $df/dx = F(x)$.



Geometrical Interpretation

According to equation $df = (df/dx)\, dx$ is the infinitesimal change in f when one goes from (x) to $(x + dx)$. The fundamental theorem says that if you chop the interval from a to b into many tiny pieces, dx, and add up the increments df from each little piece, the result is equal to the total change in f is $f(b) - f(a)$.

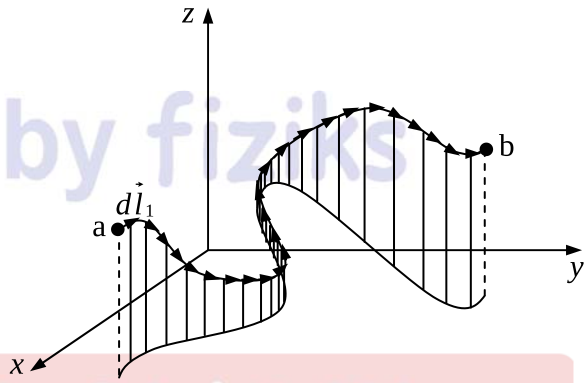
In other words, there are two ways to determine the total change in the function: either subtract the values at the ends or go step-by-step, adding up all the tiny increments as you go. You'll get the same answer either way.

1.4.3 The Fundamental Theorem for Gradients

Suppose we have a scalar function of three variables $V(x, y, z)$. Starting at point a, we moves a small distance $d\vec{l}_1$. Then

$$dV = \left(\vec{\nabla} V\right)\cdot d\vec{l}_1 .$$

Now we move a little further, by an additional small displacement $d\vec{l}_2$; the incremental change in V will be $\left(\vec{\nabla} V\right)\cdot d\vec{l}_2$. In this manner, proceeding by infinitesimal steps, we make the journey to point b. At each step we compute the gradient of V (at that point) and dot it into the displacement $d\vec{l}$ …this gives us the change in V. Evidently the total change in V in going from a to b along the path selected is



$$\int_{a \atop P}^{b} \left(\vec{\nabla} V\right)\cdot d\vec{l} = V(b) - V(a).$$

This is called the fundamental theorem for gradients; like the "ordinary" fundamental theorem, it says that the integral (here a line integral) of a derivative (here the gradient) is given by the value of the function at the boundaries (a and b).

Geometrical Interpretation

Suppose you wanted to determine the height of the Eiffel Tower. You could climb the stairs, using a ruler to measure the rise at each step, and adding them all up or you could place altimeters at the top and the bottom, and subtract the two readings; you should get the same answer either way (that's the fundamental theorem).

Corollary 1: $\int_a^b (\vec{\nabla}V)\cdot d\vec{l}$ is independent of path taken from a to b.

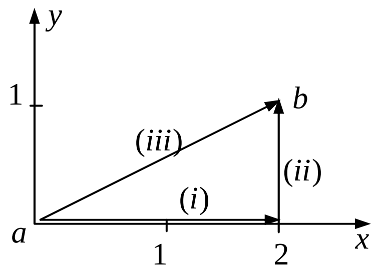
Corollary 2: $\oint (\vec{\nabla}V)\cdot d\vec{l} = 0$, since the beginning and end points are identical, and hence

$V(b) - V(a) = 0$.

Example: Let $V = xy^2$, and take point a to be the origin (0, 0, 0) and b the point (2, 1, 0). Check the fundamental theorem for gradients.

Solution: Although the integral is independent of path, we must pick a specific path in order to evaluate it. Let's go out along the x axis (step i) and then up (step ii). As always,

$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z},\ \ \vec{\nabla}V = y^2\hat{x} + 2xy\,\hat{y}$



(i) $y = 0;\ d\vec{l} = dx\,\hat{x},\ \vec{\nabla}V\cdot d\vec{l} = y^2dx = 0$, so $\int_i \vec{\nabla}V\cdot d\vec{l} = 0$

(ii) $x = 2;\ d\vec{l} = dy\,\hat{y},\ \vec{\nabla}V.d\vec{l} = 2xydy = 4y\,dy$, so

$$\int_{ii} \vec{\nabla}V\cdot d\vec{l} = \int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2$$

Evidently the total line integral is 2.

This consistent with the fundamental theorem: T(b) – T(a) =2 – 0 = 2.

Calculate the same integral along path (iii) (the straight line from a to b):

(iii) $y = \frac{1}{2}x, dy = \frac{1}{2}dx, \vec{\nabla}V\cdot d\vec{l} = y^2dx + 2xy\,dy = \frac{3}{4}x^2dx$, so

$\int_{iii} \vec{\nabla}V\cdot d\vec{l} = \int_0^2 \frac{3}{4}x^2\,dx = \frac{1}{4}x^3\Big|_0^2 = 2$. Thus the integral is independent of path.

1.4.4 The Fundamental Theorem for Divergences

The fundamental theorem for divergences states that

$$\int_V (\vec{\nabla}\cdot\vec{A})d\tau = \oint_S \vec{A}\cdot d\vec{a}$$

This theorem has at least three special names: ***Gauss's theorem, Green's theorem***, or, simply, the ***divergence theorem***. Like the other "fundamental theorems," it says that the integral of a derivative (in this case the divergence) over a region (in this case a volume) is equal to the value of the function at the boundary (in this case the surface that bounds the volume). Notice that the boundary term is itself an integral (specifically, a surface integral). This is reasonable: the "boundary" of a line is just two end points, but the boundary of a volume is a (closed) surface.

Geometrical Interpretation

If $\vec{A}$ represents the flow of an incompressible fluid, then "the flux of $\vec{A}$ (the right side of equation) is the total amount of fluid passing out through the surface, per unit time and the left side of equation shows an equal amount of liquid will be forced out through the boundaries of the region.

Example: Check the divergence theorem using the function

$$\vec{A} = y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$$

and the unit cube situated at the origin.

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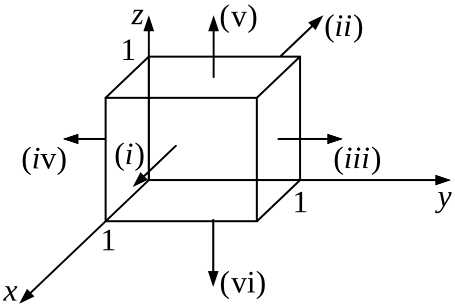
Solution: In this case



$$\vec{\nabla}\cdot\vec{A} = 2(x+y),$$

and $$\int_V 2(x+y)d\tau = 2\int_0^1\int_0^1\int_0^1 (x+y)\,dx\,dy\,dz,$$

$$\int_0^1 (x+y)dx = \frac{1}{2}+y,\ \int_0^1\left(\frac{1}{2}+y\right)dy = 1,\ \int_0^1 1dz = 1.$$

Evidently, $$\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = 2$$

To evaluate the surface integral we must consider separately the six sides of the cube:

(i) $\int \vec{A}\cdot d\vec{a} = \int_0^1\int_0^1 y^2 dy\,dz = \frac{1}{3}$ (ii) $\int \vec{A}\cdot d\vec{a} = -\int_0^1\int_0^1 y^2 dy\,dz = -\frac{1}{3}$

(iii) $\int \vec{A}\cdot d\vec{a} = \int_0^1\int_0^1 \left(2x+z^2\right)dx\,dz = \frac{4}{3}$ (iv) $\int \vec{A}\cdot d\vec{a} = -\int_0^1\int_0^1 z^2 dx\,dz = -\frac{1}{3}$

(v) $\int \vec{A}\cdot d\vec{a} = \int_0^1\int_0^1 2y\,dx\,dy = 1$ (vi) $\int \vec{A}\cdot d\vec{a} = -\int_0^1\int_0^1 0\,dx\,dy = 0$

So the total flux is:

$$\oint_S \vec{A}\cdot d\vec{a} = \frac{1}{3}-\frac{1}{3}+\frac{4}{3}-\frac{1}{3}+1+0 = 2.$$

Example: A vector field $\vec{A} = \left(\frac{5r^2}{4}\right)\hat{r}$ is given in spherical coordinates. Evaluate both sides of Divergence Theorem for the volume enclosed between

(i) $r = 1$ and $r = 2$, and

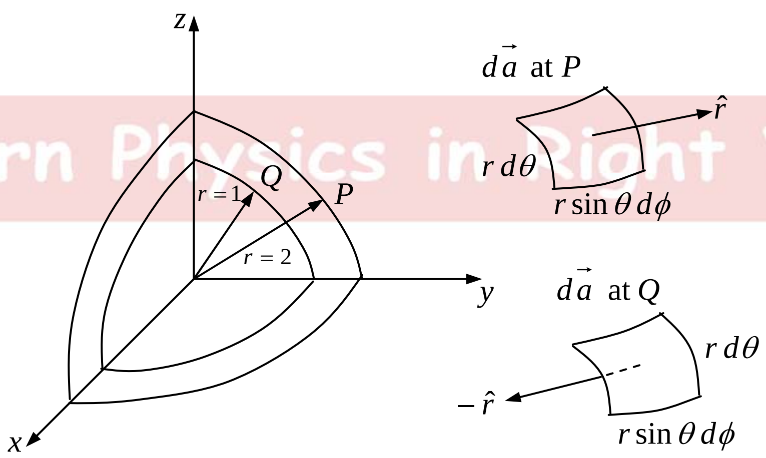
(ii) $\theta = 0$ to $\theta = \frac{\pi}{4}$ and $r = 4$.

Solution: Divergence theorem states that $\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \oint_S \vec{A}\cdot d\vec{a}$

Since $$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$A_r = \frac{5r^2}{4},\quad A_\theta = 0,\quad A_\phi = 0 \Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{5}{4}r^2\right) = 5r$$

(i)



$$L.H.S = \int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \int_v (5r)r^2\sin\theta\,dr\,d\theta\,d\phi = \int_{r=1}^{2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} 5r^3\sin\theta\,dr\,d\theta\,d\phi = 75\pi$$

R.H.S $\oint_S \vec{A}\cdot d\vec{a} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\frac{5r^2}{4}\hat{r}\right)\cdot\left(r^2\sin\theta\, d\theta\, d\phi\hat{r}\right) + \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\frac{5r^2}{4}\hat{r}\right)\cdot\left(-r^2\sin\theta\, d\theta\, d\phi\hat{r}\right)$

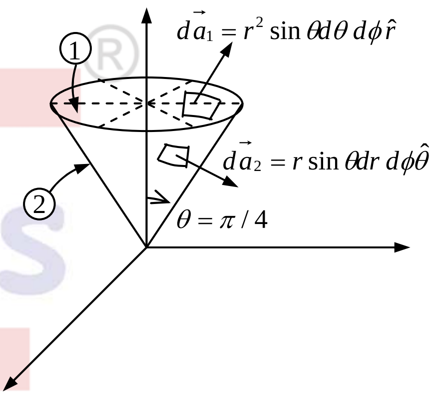
$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{5}{4}(2)^4\sin\theta\, d\theta\, d\phi - \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{5}{4}(1)^4\sin\theta\, d\theta\, d\phi = 75\pi$

So L.H.S. = R.H.S. = 75π

Divergence theorem proved.

(ii) L.H.S. of Divergence Theorem

$\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \int_{r=0}^{4}\int_{\theta=0}^{\pi/4}\int_{\phi=0}^{2\pi}(5r)\cdot r^2\sin\theta\, dr\, d\theta\, d\phi = 588.91$



R.H.S. of Divergence Theorem

$\oint_S \vec{A}\cdot d\vec{a} = \int_{S_1}\vec{A}\cdot d\vec{a} + \int_{S_2}\vec{A}\cdot d\vec{a}$

$= \int_{S_1}\left(\frac{5}{4}r^2\hat{r}\right)\cdot\left(r^2\sin\theta\, d\theta\, d\phi\hat{r}\right) + \int_{S_2}\left(\frac{5}{4}r^2\hat{r}\right)\cdot\left(r\sin\theta\, dr\, d\phi\hat{\theta}\right)$

$= \int_{\theta=0}^{\pi/4}\int_{\phi=0}^{2\pi}\frac{5}{4}(4)^4\sin\theta\, d\theta\, d\phi + 0 = 588.91$

L.H.S. = R.H.S. = 588.91 Divergence theorem proved.

1.4.5 The Fundamental Theorem for Curls

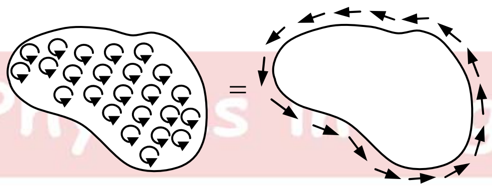
The fundamental theorem for curls, which goes by the special name of ***Stokes' theorem***, states that

$$\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \oint_P \vec{A}\cdot d\vec{l}$$

As always, the integral of a derivative (here, the curl) over a region (here, a patch of surface) is equal to the value of the function at the boundary (here, the perimeter of the patch). As in the case of the divergence theorem, the boundary term is itself an integral-specifically, a closed line integral.

Geometrical Interpretation:

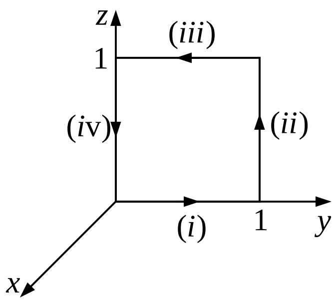
The integral of the curl over some surface (or, more precisely, the flux of the curl through that surface) represents the "total amount of swirl," and we can determine that swirl just as well by going around the edge and finding how much the flow is following the boundary (as shown in figure).



Corollary 1: $\int\left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint\left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the right side of equation vanishes.

Example: Suppose $\vec{A} = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}$. Check Stokes' theorem for the square surface shown in figure.



Solution: Here $\vec{\nabla} \times \vec{A} = (4z^2 - 2x)\hat{x} + 2z\,\hat{z}$ and $d\vec{a} = dy\,dz\,\hat{x}$

(In saying that $d\vec{a}$ points in the x direction, we are choosen to a counterclockwise line integral. We could as well write $d\vec{a} = -dy\,dz\,\hat{x}$, but then we have to go clockwise.) Since $x = 0$ for this surface,

$$\int (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \int_0^1 \int_0^1 4z^2\, dy\, dz = \frac{4}{3}$$

Now, what about the line integral? We must break this up into four segments:

(i) $x = 0, \quad z = 0, \quad \vec{A} \cdot d\vec{l} = 3y^2 dy, \qquad \int \vec{A} \cdot d\vec{l} = \int_0^1 3y^2 dy = 1,$

(ii) $x = 0, \quad y = 1, \quad \vec{A} \cdot d\vec{l} = 4z^2 dz, \int \vec{A} \cdot d\vec{l} = \int_0^1 4z^2 dz = \frac{4}{3},$

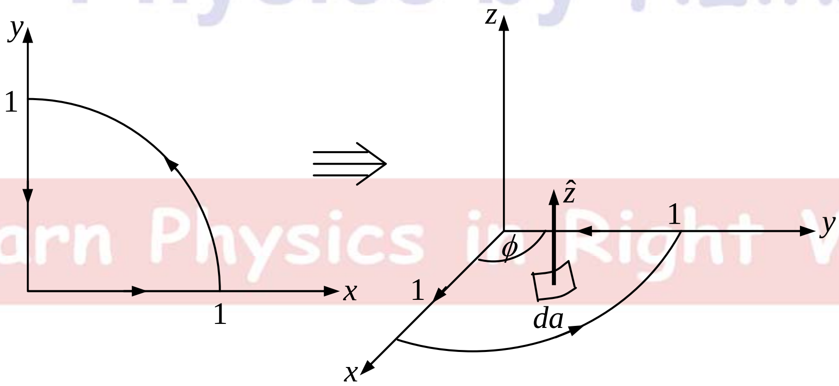
(iii) $x = 0, \quad z = 1, \quad \vec{A} \cdot d\vec{l} = 3y^2 dy, \qquad \int \vec{A} \cdot d\vec{l} = \int_1^0 3y^2 dy = -1,$

(iv) $x = 0, \quad y = 0, \quad \vec{A} \cdot d\vec{l} = 0, \qquad \int \vec{A} \cdot d\vec{l} = \int_1^0 0 dz = 0,$

So

$$\oint \vec{A} \cdot d\vec{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}.$$

Example: Given $\vec{A} = 2r\cos\phi\,\hat{r} + r\hat{\phi}$ in cylindrical coordinates. For the contour shown in figure, verify the Stokes' Theorem.



Solution: Stokes' Theorem $\int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{l}$

In cylindrical coordinates, $\vec{\nabla} \times \vec{A} = \frac{1}{r} \begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$

$A_r = 2r\cos\phi,\quad A_\phi = r\,,\, A_z = 0$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r}\left[\frac{-\partial}{\partial z}\left(r^2\right)\right]\hat{r} + \left[\frac{\partial}{\partial z}(2r\cos\phi)\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r^2\right)\frac{\partial}{\partial\phi}(2r\cos\phi)\right]\hat{z} = (2+2\sin\phi)\hat{z}$$

L.H.S.= $\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \int_{r=0}^{1}\int_{\phi=0}^{\pi/2}(2+2\sin\phi)\hat{z}\cdot\left(r\,dr\,d\phi\hat{z}\right) = \frac{\pi}{2}+1$

R.H.S. $= \oint \vec{A}\cdot d\vec{l} = \int_{r=0,1}\vec{A}\cdot d\vec{l} + \int_{\phi=0,\pi/2}\vec{A}\cdot d\vec{l} + \int_{r=1,0}\vec{A}\cdot d\vec{l}$

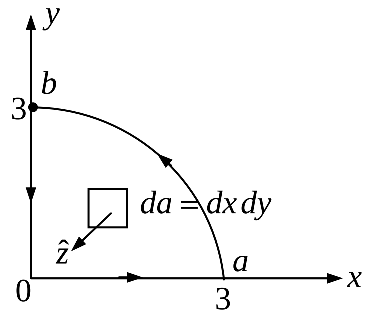
$\vec{A}\cdot d\vec{l} = \left(2r\cos\phi\hat{r} + r\hat{\phi}\right)\cdot\left(dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z}\right) = 2r\cos\phi\,dr + r^2 d\phi$

$$\oint \vec{A}\cdot d\vec{l} = \int_{\substack{r=0\\ at\,\phi=0}}^{1} 2r\cos\phi\,dr + \int_{\substack{\phi=0\\ at\,r=1}}^{\pi/2} r^2 d\phi + \int_{\substack{r=1\\ at\,\phi=\pi/2}}^{0} 2r\cos\phi\,dr \;= 1+\frac{\pi}{2}+0 = 1+\frac{\pi}{2}$$

L.H.S. = R.H.S. $=1+\frac{\pi}{2}$

Example: Given a vector field $\vec{A} = xy\hat{x} - 2x\hat{y}$. Verify stokes,' theorem over the path shown in figure.

Solution: Stokes' theorem $\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{l}$



$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2x & 0 \end{vmatrix} = -(2+x)\hat{z}$$

L.H.S. $= \int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a} = \int_{y=0}^{3}\int_{x=0}^{\sqrt{9-y^2}}\left(\vec{\nabla}\times\vec{A}\right)\cdot\left(dx\,dy\hat{z}\right)$, since

$r^2 = x^2+y^2 \;\; or \;\; x = \sqrt{9-y^2}$

$$= \int_0^3\left[\int_0^{\sqrt{9-y^2}} -(2+x)dx\,dy\right] = \int_0^3\left[\left|-2x-\frac{x^2}{2}\right|_0^{\sqrt{9-y^2}} dy\right] = \int_0^3 -\left[+2\sqrt{9-y^2}+\left(\frac{9-y^2}{2}\right)\right]dy$$

$$= -\left[y\sqrt{9-y^2} + 9\sin^{-1}\frac{y}{3} + \frac{9}{2}y - \frac{y^3}{6}\right]_0^3 = -9\left(1+\frac{\pi}{2}\right)$$

R.H.S. $= \oint \vec{A}\cdot d\vec{l} = \int_{0,a}\vec{A}\cdot d\vec{l} + \int_{a,b}\vec{A}\cdot d\vec{l} + \int_{b,0}\vec{A}\cdot d\vec{l}$

On $0a$, $y=0$; $\int \vec{A}\cdot d\vec{l} = \int -2x\,dy = 0$

On ab; $\int \vec{A}\cdot d\vec{l} = \int\left(xy\,dx - 2x\,dy\right) = \int_3^0 x\sqrt{9-x^2}\,dx - 2\int_0^3\sqrt{9-y^2}\,dy$

(Equation of quarter circle $x^2+y^2=9;\;\; 0\le x, y\le 3$)

$$\int \vec{A}\cdot d\vec{l} = -\frac{1}{3}\left(9-x^2\right)^{3/2}\Big|_3^0 - \left[y\sqrt{9-y^2} + 9\sin^{-1}\frac{y}{3}\right]_0^3 = -9\left(1+\frac{\pi}{2}\right)$$

On $b0$, $x=0$; $\int \vec{A}\cdot d\vec{l} = 0 \Rightarrow \oint \vec{A}\cdot d\vec{l} = -9\left(1+\frac{\pi}{2}\right)$ $\Rightarrow$ L.H.S. = R.H.S $= -9\left(1+\frac{\pi}{2}\right)$

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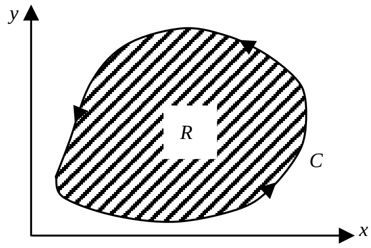
1.5 Greens Theorem



Let $\vec{f}(x,y) = P(x,y)\hat{x} + Q(x,y)\hat{y}$ is a two-dimensional vector field and R is some region in the xy-plane where C is the boundary of that region, oriented counterclockwise

"Greens theorem states that the line integral of $\vec{f}$ around the boundary of R is the same as the double integral of the curl of $\vec{f}$ within R."

$$\iint_R \left(\vec{\nabla}\times\vec{f}\right)\cdot d\vec{a} = \oint_C \vec{f}\cdot d\vec{l}$$

Note: Green's Theorem is Stokes's theorem applied to two dimensions.

$$\vec{\nabla}\times\vec{f} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & O \end{vmatrix} = \hat{z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right),\ d\vec{a} = dxdy\hat{z} \Rightarrow \iint_R \left(\vec{\nabla}\times\vec{f}\right)\cdot d\vec{a} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

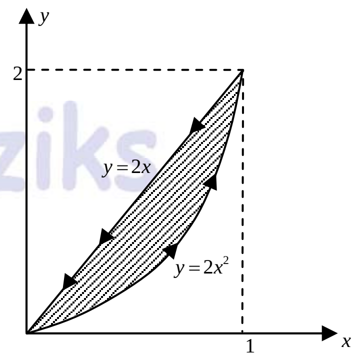
$$\oint_C \vec{f}\cdot d\vec{l} = \oint_C \left[P(x,y)dx + Q(x,y)dy\right]$$

$$\Rightarrow \oint_C \underbrace{\left[P(x,y)dx + Q(x,y)dy\right]}_{\text{Anti-clockwise}} = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]dxdy$$

If $\vec{f}$ is conservative $\oint_C \vec{f}\cdot d\vec{l} = 0 \Rightarrow \boxed{\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}}$

Example: Evaluate $I = \oint_C \left[\left(x^2 - y^2\right)dx + 2xydy\right]$ where curve is boundary of

$$R\left\{(x,y): 0 \le x \le 1, 2x^2 \le y \le 2x\right\}$$



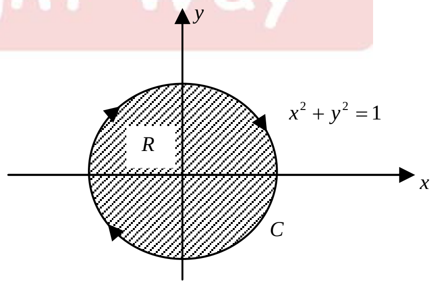
Solution: $P = \left(x^2 - y^2\right),\ Q = 2xy \Rightarrow \dfrac{\partial Q}{\partial x} = 2y, \dfrac{\partial P}{\partial y} = -2y$

$$\Rightarrow I = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]dxdy = \int_{x=0}^{1}\int_{y=2x^2}^{y=2x} \left[2y - (-2y)\right]dxdy\,,$$

$$\Rightarrow I = \int_{x=0}^{1} \left[2y^2\right]_{y=2x^2}^{y=2x} dx = \int_{x=0}^{1} \left[8x^2 - 8x^4\right]dx$$

$$\Rightarrow I = \frac{8x^3}{3} - \frac{8x^5}{5}\Bigg|_{x=0}^{1} = \frac{8}{3} - \frac{8}{5} = \frac{16}{15}$$

Example: Evaluate $I = \oint_C \left[2ydx - 3xdy\right]$ where curve is boundary of a circle as shown in figure.

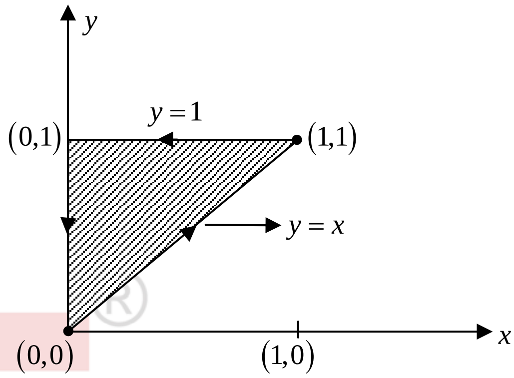


Solution: $P = 2y,\ Q = -3x \Rightarrow \dfrac{\partial Q}{\partial x} = -3, \dfrac{\partial P}{\partial y} = 2$

$$\Rightarrow I = -\iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]dxdy = -\iint_R \left[(-3-2)\,dxdy\right]$$

$$\Rightarrow I = 5\iint_R dxdy = 5\times\pi(1)^2 = 5\pi$$

Example: Evaluate $I=\oint_C \left[xdx-x^2y^2dy\right]$ where curve is boundary of a region as shown in figure.



Solution: $P=x,\ Q=-x^2y^2 \Rightarrow \frac{\partial Q}{\partial x}=-2xy^2, \frac{\partial P}{\partial y}=0$

$$\Rightarrow I=\iint_R\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]dxdy=\int_{x=0}^{1}\int_{y=x}^{y=1}\left(-2xy^2-0\right)dxdy$$

$$\Rightarrow I=\int_{x=0}^{1}\left[\frac{-2xy^3}{3}\right]_{y=x}^{y=1}dx=\int_{x=0}^{1}\left[-\frac{2x}{3}+\frac{2x^4}{3}\right]dx$$

$$\Rightarrow I=-\frac{2}{3}\cdot\frac{x^2}{2}+\frac{2}{3}\cdot\frac{x^5}{5}\Bigg|_{x=0}^{1}=-\frac{x^2}{3}+\frac{2}{15}x^5\Bigg|_{x=0}^{1}$$

$$=-\frac{1}{3}+\frac{2}{15}=-\frac{1}{5}$$

Example: $I=\oint_C\left[xydx+(x+y)dy\right]=?$ where C is the curve bounding the unit disk R.

Solution: $P=xy, Q=x+y\Rightarrow\frac{\partial Q}{\partial x}=1,\frac{\partial P}{\partial y}=x$

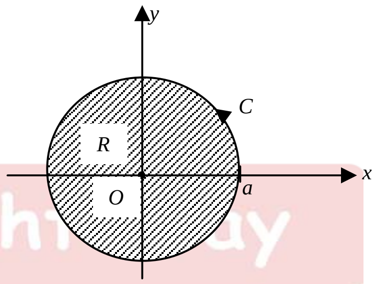
Thus $I=\iint_R\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]dxdy\Rightarrow I=\iint_R(1-x)dxdy$

Put $x=r\cos\theta, y=r\sin\theta$

$$\Rightarrow I=\int_0^{2\pi}\int_0^1(1-r\cos\theta)rdrd\theta=\int_0^{2\pi}\left[\left(\frac{r^2}{2}-\frac{r^3}{3}\cos\theta\right)\Bigg|_{r=0}^{1}\right]d\theta$$

$$\Rightarrow I=\int_0^{2\pi}\left(\frac{1}{2}-\frac{\cos\theta}{3}\right)d\theta=\left[\frac{\theta}{2}-\frac{\sin\theta}{3}\right]_0^{2\pi}=\pi$$

Example: $I=\oint\left[x^2ydx-xy^2dy\right]=?$



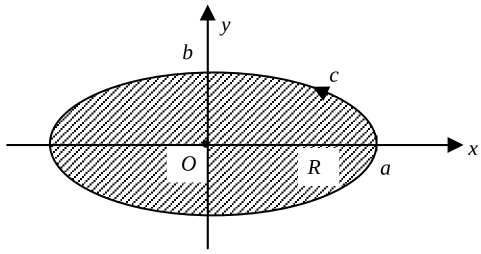
Solution: $P=x^2y, Q=-xy^2\Rightarrow\frac{\partial Q}{\partial x}=-y^2,\frac{\partial P}{\partial y}=x^2$

$$I=\iint_R\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]dxdy\Rightarrow I=\iint_R\left(-y^2-x^2\right)dxdy$$

$$\Rightarrow I=-\int_0^{2\pi}\int_0^a\left(r^2\sin^2\theta+r^2\cos^2\theta\right)rdrd\theta$$

$$\Rightarrow I=-\frac{a^4}{4}\times2\pi=-\frac{\pi a^4}{2}$$

Example: $I=\oint\left[(x+y)dx-(x-y)dy\right]=?$



Solution: $P=x+y,\ Q=-(x-y)\Rightarrow\frac{\partial Q}{\partial x}=-1,\frac{\partial P}{\partial y}=1$

$$I=\iint_R\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]dxdy=\iint_R(-1-1)dxdy\Rightarrow I=-2\iint_R dxdy$$

$$=-2\pi ab$$

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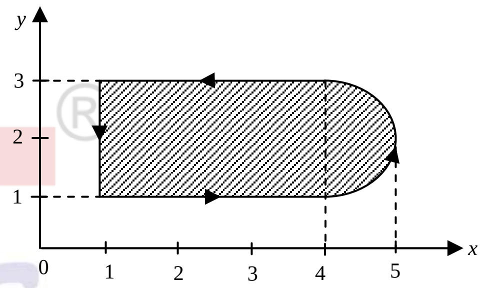
Example: Consider a unit circle C in the xy-plane, centered at the origin. The value of the integral $\oint\left[(\sin x-y)dx-(\sin y-x)dy\right]$ over the circle C, traversed anti-clockwise is

Solution: $P=\sin x-y, Q=-(\sin y-x)\Rightarrow \frac{\partial Q}{\partial x}=1, \frac{\partial P}{\partial y}=-1$

$$I=\iint_R\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]dxdy=\iint(1-(-1))dxdy=2\pi(1)^2=2\pi$$

Example: The closed curve C forms the boundary of region R shown in the figure. The region R is the area enclosed by the union of a 2×3 rectangle and semi-circle of radius 1. Then find the line integral $\int_C(xdy-ydx)=?$



Solution: $\int_C(xdy-ydx)=\int_C(-ydx+xdy)$

$$\Rightarrow P=-y, Q=x \Rightarrow \frac{\partial Q}{\partial x}=1, \frac{\partial P}{\partial y}=-1$$

$$\Rightarrow \int_C(-ydx+xdy)=\iint_R(1-(-1))dxdy=2\iint_R dxdy=2\times\left[2\times 3+\frac{\pi(1)^2}{2}\right]=12+\pi$$

1.6 The Theory of Vector Fields

If the curl of a vector field ($\vec{F}$) vanishes (everywhere), then $\vec{F}$ can be written as the gradient of a scalar potential (V): $\vec{\nabla}\times\vec{F}=0\Leftrightarrow\vec{F}=-\vec{\nabla}V$

(The minus sign is purely conventional.)

Theorem 1: Curl-less (or "irrotational") fields. The following conditions are equivalent (that is, $\vec{F}$ satisfies one if and only if it satisfies all the others):

(a) $\vec{\nabla}\times\vec{F}=0$ everywhere.

(b) $\int_a^b\vec{F}\cdot d\vec{l}$ is independent of path, for any given end points.

(c) $\oint\vec{F}\cdot d\vec{l}=0$ for any closed loop.

(d) $\vec{F}$ is the gradient of some scalar, $\vec{F}=-\vec{\nabla}V$.

The scalar potential is not unique-any constant can be added to V with impunity, since this will not affect its gradient.

If the divergence of a vector field ($\vec{F}$) vanishes (everywhere), then $\vec{F}$ can be expressed as the curl of a vector potential $(\vec{A})$:

$$\vec{\nabla}\cdot\vec{F}=0\Leftrightarrow\vec{F}=\vec{\nabla}\times\vec{A}$$

That's the main conclusion of the following theorem:

Theorem 2: Divergence-less (or "solenoidal") fields. The following conditions are equivalent:

(a) $\vec{\nabla}\cdot\vec{F}=0$ everywhere.

(b) $\int\vec{F}\cdot d\vec{a}$ is independent of surface, for any given boundary line.

(c) $\oint\vec{F}\cdot d\vec{a}=0$ for any closed surface.

(d) $\vec{F}$ is the curl of some vector, $\vec{F}=\vec{\nabla}\times\vec{A}$.

The vector potential is not unique-the gradient of any scalar function can be added to $\vec{A}$ without affecting the curl, since the curl of a gradient is zero.

Chapter 5
Fourier Series

5.1 Periodic Functions and Trigonometric Series

5.1.1 Periodic Functions

A function $f(x)$ is called periodic if it is defined for all (except for certain isolated x such as $\pm\pi/2$, $\pm 3\pi/2$......, for $\tan x$ whose period is π) real x and if there is some positive number p such that

$$f(x+p) = f(x) \qquad \text{for all } x.$$

The number p is called **period** of $f(x)$. The graph of such function is obtained by periodic repetition of its graph in any interval of length p.

Fundamental Period: If a periodic function $f(x)$ has a smallest period $p\ (>0)$, this is often called the fundamental period of $f(x)$.

NOTE: (i) Familiar periodic functions are sine and cosine functions.

(ii) The function $f = \text{constant}$ is also a periodic function.

(iii) The functions that are not periodic are x, x^2, x^3, e^x, $\cosh x$, $\ln x$ etc.

(iv) $\because f(x+2p) = f[(x+p)+p] = f(x+p) = f(x)$.

Thus, for any integer n, $f(x+np) = f(x)$. Hence $2p, 3p$,...... are also period of $f(x)$.

(v) If $f(x)$ and $g(x)$ have period p, then the function $h(x) = af(x) + bg(x)$ (a, b constants) has also period p.

Example: (i) For $\sin x$ and $\cos x$ the fundamental period is 2π.

(ii) For $\sin 2x$ and $\cos 2x$ the fundamental period is π.

(iii) For $\tan x$ and $\cot x$ the fundamental period is π.

(iv) For $\sin \pi x$ and $\cos \pi x$ the fundamental period is 2.

(v) For $\sin 2\pi x$ and $\cos 2\pi x$ the fundamental period is 2.

(vi) A function without fundamental period is $f = \text{constant}$.

5.1.2 Trigonometric Series

Let's represent various functions of period $p = 2\pi$ in terms of simple functions

$$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x, \ldots\ldots\ \cos nx,\ \sin nx, \ldots\ldots$$

These functions have period 2π. Figure below shows the first few of them.

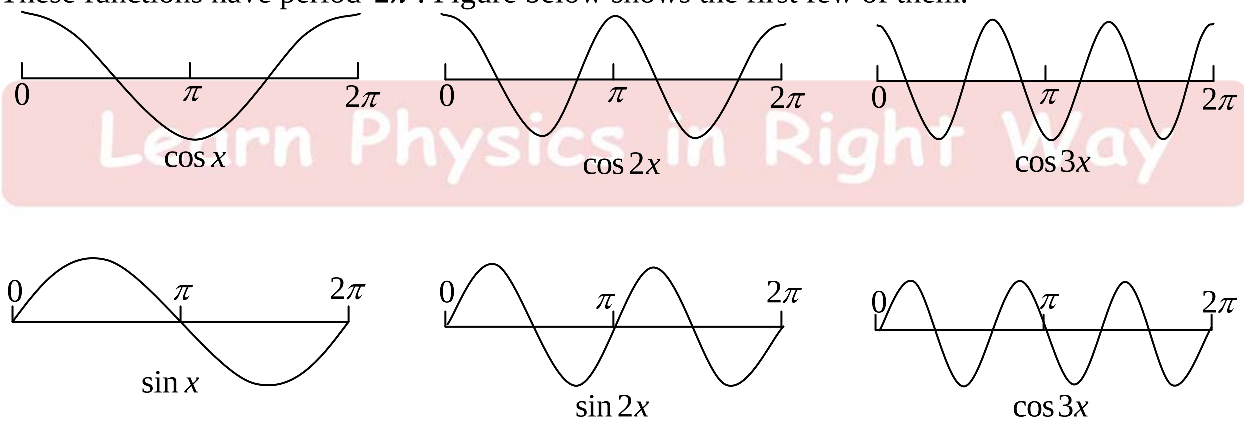


Figure: Cosine and sine functions having the period 2π

The series that will arise in this connection will be of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + ...$$

where $a_0, a_1, a_2 ... b_1, b_2 ...$ are real constants. Such a series is called trigonometric series and the a_n and b_n are called the coefficient of the series. Thus, we may write series

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

We see that each term of the series has the period 2π. Hence if the series converges, its sum will be a function of period 2π.

NOTE:

The trigonometric series can be used for representing any practically important periodic function f, simple or complicated, of any period p. This series will then be called the Fourier series of f.

5.2 Fourier Series

Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine and sine functions. These series are trigonometric series whose coefficients are determined from $f(x)$ by the "Euler Formulas".

Condition for Existence of Fourier Series

(i) $f(x)$ is periodic and single valued.

(ii) $f(x)$ is finite at all points in the given interval (i.e. it is bounded and have upper limit).

(iii) $f(x)$ may have finite number of discontinuities (i.e. it may be piece-wise continuous).

(iv) $f(x)$ may have finite number of maxima or minima or both.

Let us assume that $f(x)$ is periodic function of period 2π and is integrable over a period. Let us further assume that $f(x)$ can be represented by a trigonometric series, (assume that this series converges and has $f(x)$ as its sum)

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) \qquad ....(1)$$

5.2.1 Euler Formulas for the Fourier Coefficients

Determination of constant a_0***:***

From equation (1), we have

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) \right] dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right)$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0 \qquad \because \int_{-\pi}^{\pi} \cos nx dx = 0, \quad \int_{-\pi}^{\pi} \sin nx dx = 0$$

Thus,

$$\boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx} \qquad ....(2)$$

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Determination of constant a_n :

Multiply equation (1) by $\cos mx$, where m is any fixed positive integer, then

$$\int_{-\pi}^{\pi} f(x)\cos mx dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mx dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos mx dx = a_0\int_{-\pi}^{\pi}\cos mx dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos mx dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx dx\right)$$

$$\because \int_{-\pi}^{\pi}\cos mx dx = 0$$

$$\int_{-\pi}^{\pi}\cos nx\cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)x dx = \begin{cases}0, & m \neq n\\ \pi, & m = n\end{cases}$$

$$\int_{-\pi}^{\pi}\sin nx\cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi}\sin(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi}\sin(n-m)x dx = 0 \text{, always}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos nx dx = a_n\pi \text{, when } n = m$$

Thus,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \qquad \text{....(3)}$$

Determination of constant b_n :

Multiply equation (1) by $\sin mx$, where m is any fixed positive integer, then

$$\int_{-\pi}^{\pi} f(x)\sin mx dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\sin mx dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\sin mx dx = a_0\int_{-\pi}^{\pi}\sin mx dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\sin mx dx + b_n\int_{-\pi}^{\pi}\sin nx\sin mx dx\right)$$

$$\int_{-\pi}^{\pi}\cos nx\sin mx dx = \frac{1}{2}\int_{-\pi}^{\pi}\sin(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi}\sin(m-n)x dx = 0 \text{, always., } \int_{-\pi}^{\pi}\sin mx dx = 0$$

$$\int_{-\pi}^{\pi}\sin nx\sin mx dx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)x dx - \frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)x dx = \begin{cases}0, & m \neq n\\ \pi, & m = n\end{cases}.$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\sin nx dx = b_n\pi \text{ when } n = m;$$

Thus,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \qquad \text{....(4)}$$

5.2.2 Orthogonality of the Trigonometric System

The trigonometric system

1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$,…….. $\cos nx$, $\sin nx$,……..

is orthogonal on the interval $-\pi \le x \le \pi$ (hence on any interval of length 2π, because of periodicity). Thus, for any integer m and n we have

$$\int_{-\pi}^{\pi}\cos mx\cos nx dx = 0 \qquad (m \neq n)$$

and

$$\int_{-\pi}^{\pi}\sin mx\sin nx dx = 0 \qquad (m \neq n)$$

and for any integer m and n (including $m = n$) we have $\int_{-\pi}^{\pi}\cos mx\sin nx dx = 0$

5.2.3 Convergence and Sum of Fourier Series

If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \le x \le \pi$ and has left hand and right hand derivative at each point of that interval, then the Fourier series of $f(x)$ with coefficient a_0, a_n, b_n is convergent.

Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left and right-hand limit of $f(x)$ at x_0.

NOTE: (i) The left-hand limit of $f(x)$ at x_0 is $f(x_0-0)=\lim\limits_{h\to 0} f(x_0-h)$.

The right-hand limit of $f(x)$ at x_0 is $f(x_0+0)=\lim\limits_{h\to 0} f(x_0+h)$.

(ii) Function $f(x)$ is continuous at x_0, if

$$f(x_0-0)=f(x_0+0)=f(x_0)$$

(iii) The left-hand derivative of $f(x)$ at x_0 is $\lim\limits_{h\to 0}\dfrac{f(x_0-h)-f(x_0)}{-h}$.

The right-hand derivative of $f(x)$ at x_0 is $\lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}$.

Function $f(x)$ is differentiable at x_0, if $L.H.D.=R.H.D.$

Some Basic Mathematics Result to be use in Fourier Series

(i) $\int u(x)v(x)dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$

where $u'=\dfrac{du}{dx}, u''=\dfrac{d^2u}{dx^2},\ldots$ and $v_1=\int v dx,\ v_2=\int v_1 dx, \ldots$

Example: $\int x^3 \sin nx dx = x^3\left(-\dfrac{\cos nx}{n}\right)-3x^2\left(-\dfrac{\sin nx}{n^2}\right)+6x\left(\dfrac{\cos nx}{n^3}\right)-6\left(\dfrac{\sin nx}{n^4}\right)$

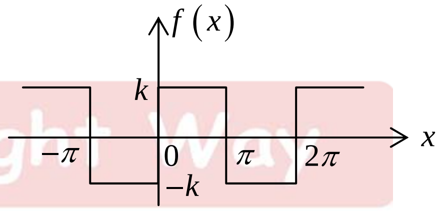
(i) $\cos n\pi = (-1)^n,\ n=0,1,2\ldots,\qquad \sin n\pi = 0,\ n=0,1,2\ldots$

$$\cos n\frac{\pi}{2}=\begin{cases}0, & n=1,3,5,\ldots\\ 1, & n=0,4,8\ldots\\ -1, & n=2,6,10\ldots\end{cases},\qquad \sin n\frac{\pi}{2}=\begin{cases}0, & n=0,2,4\ldots\\ 1, & n=1,5,9\ldots\\ -1, & n=3,7,11\ldots\end{cases}$$

Example: Find the Fourier coefficient of the periodic function $f(x)$ as shown in figure:

$$f(x)=\begin{cases}-k & \text{if } -\pi<x<0\\ k & \text{if } 0<x<\pi\end{cases} \quad \text{and} \quad f(x+2\pi)=f(x)$$

Hence show that: $1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\ldots=\dfrac{\pi}{4}$



Solution: Let $f(x)=a_0+\sum\limits_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$

$$\because a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$\Rightarrow a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}(-k)\,dx+\int_{0}^{\pi}(k)\,dx\right]=\frac{1}{2\pi}\left[[-kx]_{-\pi}^{0}+[kx]_{0}^{\pi}\right]=0$$

This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ to π is zero.

$$\because a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx dx + \int_{0}^{\pi}(k)\cos nx dx\right] = \frac{1}{\pi}k\left[-\left\{\frac{\sin nx}{n}\right\}_{-\pi}^{0} + \left\{\frac{\sin nx}{n}\right\}_{0}^{\pi}\right] = 0$$

$$\because b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx dx + \int_{0}^{\pi}(k)\sin nx dx\right]$$

$$\Rightarrow b_n = \frac{1}{\pi}k\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{0} - \left\{\frac{\cos nx}{n}\right\}_{0}^{\pi}\right] = \frac{1}{\pi}k\left[\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n}\right] = \frac{1}{\pi}k\left[\frac{2}{n} - \frac{2(-1)^n}{n}\right]$$

If n is even $b_n = 0$ and if n is odd $b_n = \frac{4k}{n\pi}$.

Thus, Fourier series is $f(x) = \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]$

The partial sums are $S_1 = \frac{4k}{\pi}\sin x$, $S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$, etc

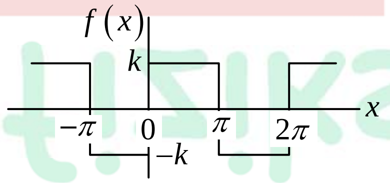


Figure (a): The given function $f(x)$ (Period square wave)

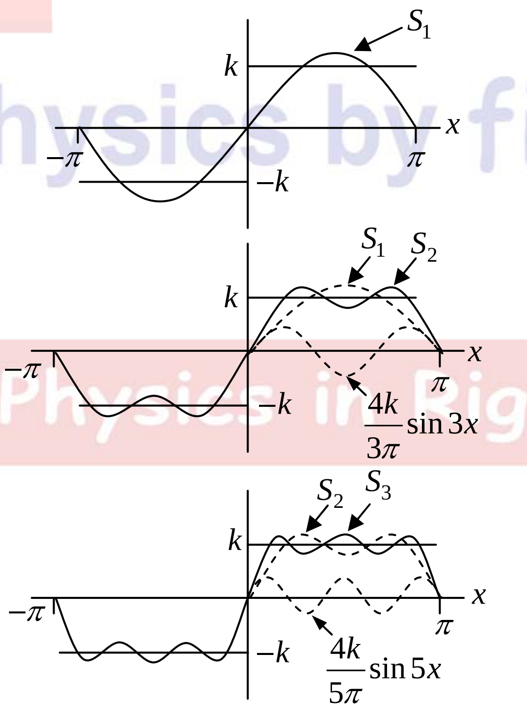


Figure (b): The first three partial sums of the corresponding Fourier series

NOTE: The above graph seems to indicate that the series is convergent and has the sum $f(x)$, the given function. Notice that at $x=0$ and $x=\pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the values $-k$ and $+k$ of our function.

Assuming that $f(x)$ is the sum of the series and setting $x=\frac{\pi}{2}$, we have

$$f\left(\frac{\pi}{2}\right)=k=\frac{4k}{\pi}\left[\sin\frac{\pi}{2}+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\sin\frac{5\pi}{2}+\ldots\right]$$

$$\Rightarrow 1=\frac{4}{\pi}\left[1+\frac{1}{3}(-1)+\frac{1}{5}(1)+\frac{1}{7}(-1)+\ldots\right]=\frac{4}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right]\Rightarrow\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

Example: The square wave in previous example has a jump at $x=0$. Its left-hand limit there is $-k$ and its right-hand limit there is $+k$. Hence average of these limits is 0. Thus, Fourier series converge to this value at $x=0$, because then all its terms are 0. Similarly, for the other jump we can verify this.

Example: Find the Fourier coefficient of the periodic function $f(x)$:

$$f(x)=x,\quad -\pi<x<\pi \qquad \text{having period } 2\pi$$

Solution: $\because a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx\Rightarrow a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}xdx=0$

$$\because a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx\ \Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos nxdx=\frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right)-1\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}=0$$

$$\because b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nxdx=\frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-1\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$\Rightarrow b_n=\frac{1}{\pi n}\left[-\pi\cos n\pi-\pi\cos n\pi\right]=-\frac{2}{n}(-1)^n$$

Thus $f(x)=-2\left[-\frac{\sin x}{1}+\frac{\sin 2x}{2}-\frac{\sin 3x}{3}+\ldots\right]=2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\ldots\right]$

Example: Find the Fourier coefficient of the periodic function $f(x)$:

$$f(x)=x^2,\quad -\pi<x<\pi \qquad \text{having period } 2\pi$$

Solution: $\because a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx\Rightarrow a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx=\frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}=\frac{1}{6\pi}\left[\pi^3+\pi^3\right]=\frac{\pi^2}{3}$

$$\because a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nxdx=\frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-2x\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$\Rightarrow a_n=\frac{2}{n^2\pi}\left[\pi\cos n\pi+\pi\cos n\pi\right]=\frac{4}{n^2}(-1)^n$$

$$\because b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

$$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nxdx=\frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right)-2x\left(-\frac{\sin nx}{n^2}\right)+\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}=0$$

Thus $f(x)=\frac{\pi^2}{3}-4\left(\cos x-\frac{1}{4}\cos 2x+\frac{1}{9}\cos 3x-\ldots.\right)$

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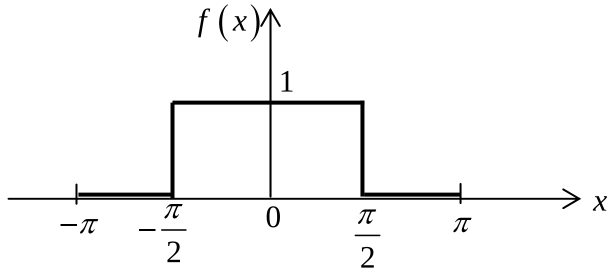
Example: Find the Fourier series of the periodic function $f(x)$ as shown in figure:



Solution: The given function is periodic in 2π .

$$\because a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \Rightarrow a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2}(0)\,dx + \int_{-\pi/2}^{\pi/2}(1)\,dx\right] = \frac{1}{2\pi}\times\pi = \frac{1}{2}$$

$$\because a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(0)\cos nx dx + \int_{-\pi/2}^{\pi/2}(1)\cos nx dx\right] = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi/2}^{\pi/2} = \frac{2}{n\pi}\sin n\frac{\pi}{2}$$

$$\because b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \Rightarrow b_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(0)\sin nx dx + \int_{-\pi/2}^{\pi/2}(1)\sin nx dx\right] = \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\pi/2}^{\pi/2} = 0$$

$$\Rightarrow b_n = \frac{1}{\pi}k\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{0} - \left\{\frac{\cos nx}{n}\right\}_{0}^{\pi}\right] = \frac{1}{\pi}k\left[\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n}\right] = \frac{1}{\pi}k\left[\frac{2}{n} - \frac{2(-1)^n}{n}\right]$$

$$\because \sin n\frac{\pi}{2} = \begin{cases} 0, & n = 2,4,6... \\ 1, & n = 1,5,9... \\ -1, & n = 3,7,11... \end{cases}$$

Thus, Fourier series is $f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + ...\right]$

Example: Find the Fourier coefficient of the periodic function $f(x)$:

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0\ , & 0 < x < \pi \end{cases} \quad \text{having period } 2\pi$$

Solution: $\because a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \Rightarrow a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(1)\,dx + \int_{0}^{\pi}(0)\,dx\right] = \frac{1}{2\pi}\times\pi = \frac{1}{2}$

$$\because a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(1)\cos nx dx + \int_{0}^{\pi}(0)\cos nx dx\right] = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} = 0$$

$$\because b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \Rightarrow b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(1)\sin nx dx + \int_{0}^{\pi}(0)\sin nx dx\right] = \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0}$$

$$\Rightarrow b_n = -\frac{1}{n\pi}[1 - \cos n\pi] = -\frac{1}{n\pi}\left[1 - (-1)^n\right] = -\frac{2}{n\pi},\ n = 1,3,.. \text{ and } 0,\ n = 2,4,..$$

Thus, Fourier series is

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right]$$

Example: Find the Fourier coefficient of the periodic function $f(x)$:

$$f(x) = x, \quad 0 < x < 2\pi \qquad \text{having period } 2\pi$$

Solution: $\because a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} xdx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi}\times\frac{4\pi^2}{2} = \pi$

$$\because a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx \Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nxdx = \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = 0$$

$$\because b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx \Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nxdx = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$\Rightarrow b_n = \frac{1}{\pi n}\left[-2\pi\cos 2n\pi + 0\right] = -\frac{2}{n}$$

$$\because f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right), \quad f(x) = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x \ldots\right]$$

Example: Find the Fourier coefficient of the periodic function $f(x)$:

$$f(x) = x^2, \quad 0 < x < 2\pi \qquad \text{having period } 2\pi$$

Solution: $\because a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{6\pi}\times 8\pi^3 = \frac{4}{3}\pi^2$

$$\because a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nxdx = \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$\Rightarrow a_n = \frac{2}{n^2\pi}\left[2\pi\cos 2n\pi + 0\right] = \frac{2}{n^2\pi}\times 2\pi = \frac{4}{n^2}$$

$$\because b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nxdx = \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + \left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi} = -\frac{4\pi}{n}$$

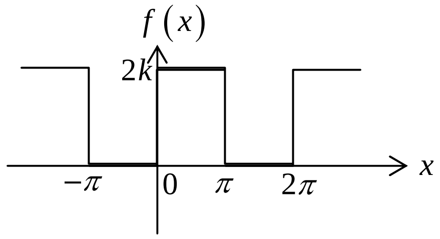
$$f(x) = \frac{4}{3}\pi^2 + 4\left(\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots\right) - 4\pi\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right)$$

Sum and Scalar Multiple

(a) The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

(b) The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

Example: Find the Fourier series of the periodic function $f(x)$ as shown in figure:



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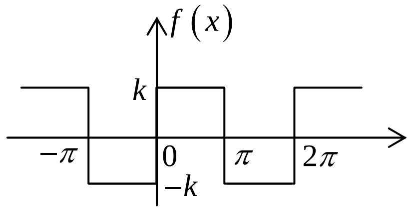
Solution: We have already calculated the Fourier series of the periodic function $f(x)$ as shown in figure.

The Fourier series is $f(x)=\frac{4k}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...\right]$.



The function given in the problem can be obtained by adding k to the above function. Thus, the Fourier series of a sum $k+f(x)$ are the sums of the corresponding Fourier series of k and $f(x)$.

The Fourier series is $f(x)=k+\frac{4k}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...\right]$.

Example: Find the Fourier series of the periodic function

$$f(x)=x+\pi \; ; \quad (-\pi<x<\pi) \text{ having period } 2\pi$$

Solution: Let $f(x)=f_1+f_2$, where $f_1=x$ and $f_2=\pi$.

The Fourier coefficient of $f_2=\pi$ is $a_0=\pi$, $a_n=0$ and $b_n=0$.

The Fourier coefficient of $f_1=x$ is $a_0=0$, $a_n=0$ and $b_n=-2\frac{(-1)^n}{n}$; $n=1,2,3....$.

Thus, the Fourier series of $f(x)$ is $f(x)=\pi+2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x...\right]$.

Example: Find the Fourier series of the periodic function

$$f(x)=x+x^2 \; ; \quad (-\pi<x<\pi) \text{ having period } 2\pi$$

Solution: Let $f(x)=f_1+f_2$, where $f_1=x$ and $f_2=x^2$.

Thus $f_1(x)=2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-...\right]$, $f_2(x)=\frac{\pi^2}{3}-4\left(\cos x-\frac{1}{4}\cos 2x+\frac{1}{9}\cos 3x-....\right)$

Thus, the Fourier series of $f(x)$ is

$$f(x)=\frac{\pi^2}{3}-4\left(\cos x-\frac{1}{4}\cos 2x+\frac{1}{9}\cos 3x-....\right)+2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-...\right].$$

Example: Find the Fourier series of the periodic function

$$f(x)=x+x^2 \; ; \quad (0<x<2\pi) \text{ having period } 2\pi$$

Solution: Let $f(x)=f_1+f_2$, where $f_1=x$ and $f_2=x^2$.

Thus $f_1(x)=\pi-2\left[\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\frac{1}{4}\sin 4x...\right]$,

$$f_2(x)=\frac{4}{3}\pi^2+4\left(\cos x+\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x-....\right)-4\pi\left(\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+....\right)$$

Thus, the Fourier series of $f(x)$ is $f(x)=f_1(x)+f_2(x)$

5.3 Function of Any Period $p = 2L$

The functions considered so far had period 2π, for simplicity. The transition from $p = 2\pi$ to $p = 2L$ is quite simple. It amounts to a stretch (or contraction) of scale on the axis.

Fourier series of a function $f(x)$ of period $p = 2L$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right) \qquad \text{....(1)}$$

where Fourier coefficients of $f(x)$ are

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

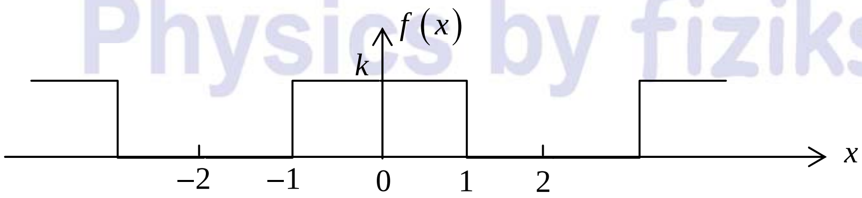
Let $v = \frac{\pi x}{L} \Rightarrow x = \frac{Lv}{\pi}$ and $dv = \frac{\pi dx}{L}$. Also $x = \pm L$ corresponds to $v = \pm\pi$.

Thus $f(x) = g(v)$ has period 2π.

Hence, we can verify that $g(v) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)\,dv$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\,dv$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\sin nv\,dv$

Example: Find the Fourier series of the function $f(x)$ as shown in figure:



$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1 \\ k & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < 2 \end{cases} \qquad p = 2L = 4,\ L = 2$$

Solution: Let $f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$

$$\because a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$\Rightarrow a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} k\,dx = \frac{k}{2}$$

$$\because a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$\Rightarrow a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}dx$$

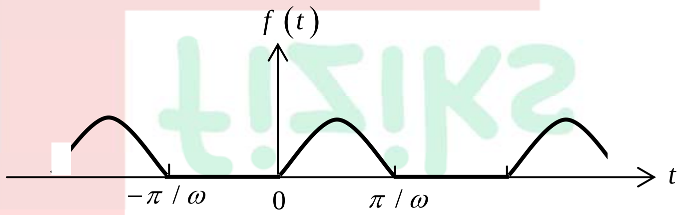
$$\Rightarrow a_n = \frac{k}{2}\left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_{-1}^{1} = \frac{2k}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 0, & n = 2,4,6..... \\ \frac{2k}{n\pi}, & n = 1,5,9..... \\ -\frac{2k}{n\pi}, & n = 3,7,11..... \end{cases}$$

$$\because b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$\Rightarrow b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\sin\frac{n\pi x}{2}dx = \frac{k}{2}\left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_{-1}^{1} = 0$$

Thus Fourier series is $f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - +....\right)$

Example: A sinusoidal voltage $E_0 \sin\omega t$ where t is time is passed through half-wave rectifier that clips the negative portion of the wave. Find the Fourier series of the resulting periodic function



$$f(t) = \begin{cases} 0 & \text{if} \quad -\frac{\pi}{\omega} < t < 0 \\ E_0 \sin\omega t & \text{if} \quad 0 < t < \frac{\pi}{\omega} \end{cases} \qquad p = 2L = 2\frac{\pi}{\omega},\ L = \frac{\pi}{\omega}$$

Solution: Let $f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right)$

$$\because a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

$$\Rightarrow a_0 = \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\,dt = \frac{\omega}{2\pi}\int_{0}^{\pi/\omega} E_0\sin\omega t dt = \frac{\omega}{2\pi}\left[-\frac{E_0}{\omega}\cos\omega t\right]_0^{\pi/\omega} = \frac{E_0}{\pi}$$

$$\because a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

$$\Rightarrow a_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\cos n\omega t dt = \frac{\omega}{\pi}\int_{0}^{\pi/\omega} E_0\sin\omega t\cos n\omega t dt = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega} 2\sin\omega t\cos n\omega t dt$$

$$\Rightarrow a_n = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt$$

For $n = 1$,

$$a_1 = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega}\sin 2\omega t dt = \frac{E_0\omega}{2\pi}\left[\frac{-\cos 2\omega t}{2\omega}\right]_0^{\pi/\omega} = 0$$

For $n = 2,3,4....$,

$$\Rightarrow a_n = \frac{E_0\omega}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$\Rightarrow a_n = \frac{E_0}{2\pi}\left[\frac{-\cos(1+n)\pi+1}{(1+n)} + \frac{-\cos(1-n)\pi+1}{(1-n)}\right] = \begin{cases} 0 & n = 3,5,7... \\ \dfrac{2E_0}{\pi(1-n^2)} & n = 2,4,6.... \end{cases}$$

$$\because b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$\Rightarrow b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\sin n\omega t dt = \frac{\omega}{\pi}\int_0^{\pi/\omega} E_0 \sin\omega t \sin n\omega t dt = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega} 2\sin\omega t\sin n\omega t dt$$

$$\Rightarrow b_n = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega}\left[-\cos(1+n)\omega t + \cos(1-n)\omega t\right]dt$$

For $n=1$, $b_1 = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega}[1-\cos 2\omega t]dt = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega}\left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^{\pi/\omega} = \frac{E_0\omega}{2\pi}\frac{\pi}{\omega} = \frac{E_0}{2}$

For $n = 2,3,4....,$

$$\Rightarrow b_n = \frac{E_0\omega}{2\pi}\left[-\frac{\sin(1+n)\omega t}{(1+n)\omega} + \frac{\sin(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega} = \frac{E_0}{2\pi}\left[\frac{-\sin(1+n)\pi}{(1+n)} + \frac{\sin(1-n)\pi}{(1-n)}\right] = 0$$

Thus, Fourier series

$$f(t) = a_0 + b_1\sin\omega t + \sum_{n=2,4...}^{\infty} a_n\cos n\omega t = a_0 + b_1\sin\omega t + \sum_{n=2,4...}^{\infty}\frac{2E_0}{\pi(1-n^2)}\cos n\omega t$$

$$\Rightarrow f(t) = \frac{E_0}{\pi} + \frac{E_0}{2}\sin\omega t - \frac{2E_0}{\pi}\left[\frac{\cos 2\omega t}{3} + \frac{\cos 4\omega t}{15} + ...\right]$$

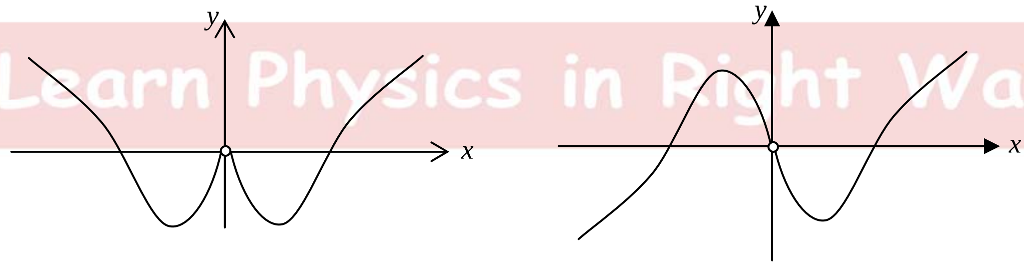
$$\Rightarrow f(t) = \frac{E_0}{\pi} + \frac{E_0}{2}\sin\omega t - \frac{2E_0}{\pi}\left[\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + ...\right]$$

5.4 Even and Odd functions and Half-Range Expansion

5.4.1 Even and Odd function

A function $g(x)$ is said to be even if $g(-x) = g(x)$, so that its graph is symmetrical with respect to vertical axis.

A function $h(x)$ is said to be even if $h(-x) = -h(x)$.



(a) Even function (b) Odd function

Since the definite integral of a function gives the area under the curve of the function between the limits of integration, we have

$$\int_{-L}^{L} g(x)dx = 2\int_0^L g(x)dx \qquad \text{for even } g(x)$$

$$\int_{-L}^{L} h(x)dx = 0 \qquad \text{for odd } h(x)$$

Fourier Cosine Series and Fourier Sine Series

Fourier series of an **even** function of period $2L$, is a **"Fourier cosine series"**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \qquad \ldots(1)$$

with coefficients (note integration from 0 to L)

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad , \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad \ldots(2)$$

Fourier series of an odd function of period $2L$, is a **"Fourier sine series"**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \qquad \ldots(3)$$

with coefficients $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

NOTE:

(i) For even function $f(x)$;

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0$.

(ii) For odd function $f(x)$;

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 0, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = 0$$

and $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

The Case of Period 2π

If $L = \pi$ and $f(x)$ is even function, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(1')$$

with coefficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \quad , \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad \ldots(2')$$

If $f(x)$ is odd function then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(3')$$

with coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \qquad \ldots(4')$$

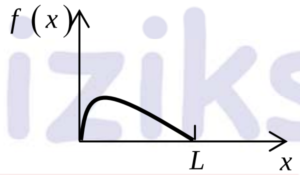
5.4.2 *Half-Range Expansion*

Half-range expansions are Fourier series. The idea is simple and useful. We could extend $f(x)$ as a function of period L and develop the extended function into a Fourier series. But this series would in general contain both cosine and sine terms.

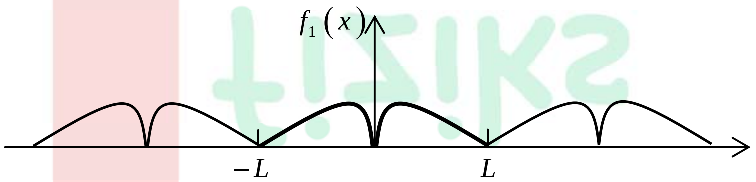
We can do better and get simpler series. For our given function $f(x)$, we can calculate **Fourier cosine series** coefficient (a_0 and a_n). This is the even periodic extension $f_1(x)$ of $f(x)$ in figure (b).

For our given function $f(x)$ we can calculate **Fourier sine series** coefficient (b_n). This is the odd periodic extension $f_2(x)$ of $f(x)$ in figure (c).

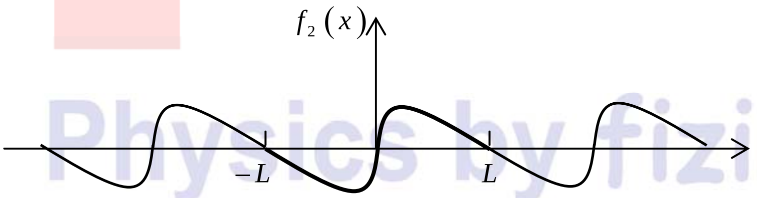
Both extensions have period $2L$. Note that $f(x)$ is given only on half the range, half the interval of periodicity of length $2L$.



(a) The given function



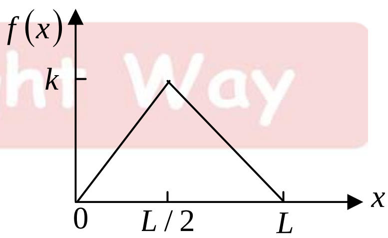
(b) $f(x)$ extended as an even periodic function of period $2L$



(c) $f(x)$ extended as an odd periodic function of period $2L$

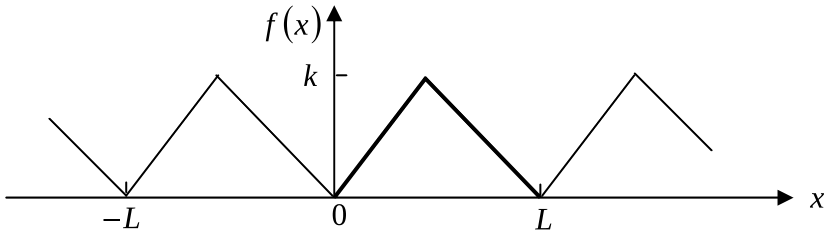
Example: Find the two half-range expansion of the function $f(x)$ as shown in figure below.

$$f(x)=\begin{cases}\dfrac{2k}{L}x & \text{if} \quad 0<x<\dfrac{L}{2}\\ \dfrac{2k}{L}(L-x) & \text{if} \quad \dfrac{L}{2}<x<L\end{cases}$$



Solution:

Even periodic extension



$$\because a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$\Rightarrow a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\,dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\,dx\right] = \frac{k}{2}$$

$$\because a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \Rightarrow a_n = \frac{1}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\cos\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\cos\frac{n\pi x}{L}dx\right]$$

Let us calculate the integral

$$\int_0^{\frac{L}{2}} x\cos\frac{n\pi x}{L}dx = \frac{L}{n\pi}\left[x\sin\frac{n\pi x}{L}\right]_0^{L/2} + \left(\frac{L}{n\pi}\right)^2\left[\cos\frac{n\pi x}{L}\right]_0^{L/2}$$

$$\Rightarrow \int_0^{\frac{L}{2}} x\cos\frac{n\pi x}{L}dx = \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

and $\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}dx = \frac{L}{n\pi}\left[(L-x)\sin\frac{n\pi x}{L}\right]_{L/2}^{L} - \left(\frac{L}{n\pi}\right)^2\left[\cos\frac{n\pi x}{L}\right]_{L/2}^{L}$

$$\Rightarrow \int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}dx = -\frac{L^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$\Rightarrow a_n = \frac{1}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\cos\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\cos\frac{n\pi x}{L}dx\right] = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

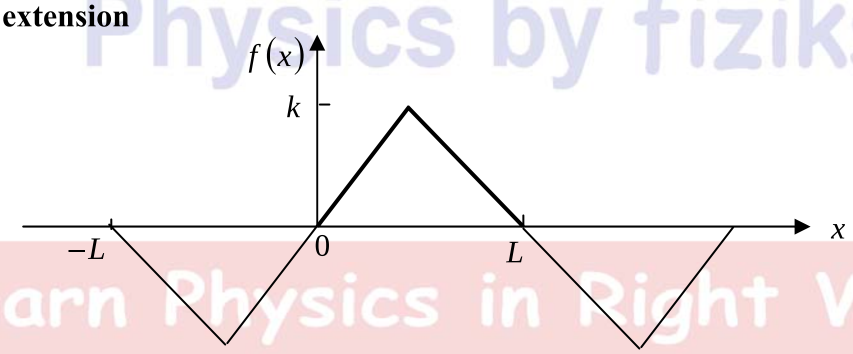
$\Rightarrow a_2 = -\frac{16k}{2^2\pi^2}, a_6 = -\frac{16k}{6^2\pi^2}, a_{10} = -\frac{16k}{10^2\pi^2}$,……….. and $a_n = 0$ if $n \neq 2, 6, 10.....$

Hence the first half-range expansion of $f(x)$ is

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{L} + \frac{1}{6^2}\cos\frac{6\pi x}{L} + \frac{1}{10^2}\cos\frac{10\pi x}{L} + ...\right)$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$ as shown in figure.

Odd periodic extension



$$\because b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \Rightarrow b_n = \frac{2}{L}\int_0^{\frac{L}{2}} f(x)\sin\frac{n\pi x}{L}dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$\Rightarrow b_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\sin\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\sin\frac{n\pi x}{L}dx\right]$$

Let us calculate the integral

$$\int_0^{\frac{L}{2}} x\sin\frac{n\pi x}{L}dx = \frac{L}{n\pi}\left[-x\cos\frac{n\pi x}{L}\right]_0^{L/2} + \left(\frac{L}{n\pi}\right)^2\left[\sin\frac{n\pi x}{L}\right]_0^{L/2}$$

$$\Rightarrow \int_0^{\frac{L}{2}} x\cos\frac{n\pi x}{L}dx = -\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

and $\int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}dx = \frac{L}{n\pi}\left[-(L-x)\cos\frac{n\pi x}{L}\right]_{L/2}^{L} - \left(\frac{L}{n\pi}\right)^2\left[\sin\frac{n\pi x}{L}\right]_{L/2}^{L}$

$$\Rightarrow \int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}dx = \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\Rightarrow b_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\sin\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\sin\frac{n\pi x}{L}dx\right] = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence the other half-range expansion of $f(x)$ is

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi x}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L} + \frac{1}{5^2}\sin\frac{5\pi x}{L} - +.....\right)$$

This Fourier sine series represents the odd periodic extension of the given function $f(x)$, of period $2L$ as shown in figure.

5.5 Complex Fourier Series

The Fourier series $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ ....(1)

can be written in complex form, which sometimes simplifies calculations.

$\because e^{inx} = \cos nx + i\sin nx$ and $e^{-inx} = \cos nx - i\sin nx$

$\Rightarrow \cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right)$ and $\sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$

Thus $a_n\cos nx + b_n\sin nx = \frac{1}{2}a_n\left(e^{inx} + e^{-inx}\right) + \frac{1}{2i}b_n\left(e^{inx} - e^{-inx}\right)$

$$\Rightarrow a_n\cos nx + b_n\sin nx = \frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx}$$

Lets take $a_0 = c_0$, $\frac{1}{2}(a_n - ib_n) = c_n$ and $\frac{1}{2}(a_n + ib_n) = k_n$, then (1) becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{inx} + k_n e^{-inx}\right) \quad ....(2)$$

where coefficients c_0, c_n and k_n are given by $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$k_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i\sin nx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx$$

We can also write (2) as (take $k_n = c_{-n}$)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx\,; \quad n = 0, \pm 1, \pm 2.......$$

This is so called complex form of the Fourier series or, complex Fourier series of $f(x)$.

Here c_n are called complex Fourier series coefficients of $f(x)$.

For a function of period $2L$

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}} \quad \text{where } c_n=\frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi x}{L}}dx\,; \quad n=0,\pm1,\pm2.......$$

Example: Find the complex Fourier series of the periodic function

$f(x)=e^x;\quad (-\pi<x<\pi)$, having period 2π.

Solution: Let Fourier series is $f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx}$

$$\because c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \Rightarrow c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}dx=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}dx=\frac{1}{2\pi}\frac{1}{(1-in)}\left[e^{(1-in)x}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{2\pi}\frac{1}{(1-in)}\left[e^{(1-in)\pi}-e^{-(1-in)\pi}\right]$$

$$\Rightarrow c_n=\frac{1}{2\pi}\frac{1}{(1-in)}\left[e^{\pi}e^{-in\pi}-e^{-\pi}e^{in\pi}\right]=\frac{1}{2\pi}\left(\frac{1+in}{1+n^2}\right)\left[e^{\pi}-e^{-\pi}\right](-1)^n \qquad \because e^{\pm in\pi}=(-1)^n$$

$$\Rightarrow c_n=\left(\frac{1+in}{1+n^2}\right)\frac{\sinh\pi}{\pi}(-1)^n \qquad \because \sinh\pi=\frac{e^{\pi}-e^{-\pi}}{2}$$

Thus, Fourier series is $f(x)=\dfrac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\left(\dfrac{1+in}{1+n^2}\right)e^{inx}$ ....(1)

Let us derive the real Fourier series

$\because (1+in)e^{inx}=(1+in)(\cos nx+i\sin nx)=(\cos nx-n\sin nx)+i(\cos nx+\sin nx)$

$\because n$ varies from $-\infty$ to $+\infty$, equation (1) has corresponding term with $-n$ instead of n.

Thus, $\because (1-in)e^{-inx}=(1-in)(\cos nx-i\sin nx)=(\cos nx-n\sin nx)-i(\cos nx+\sin nx)$

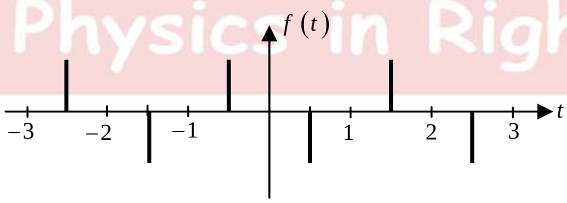
Let's add these two expressions; $(1+in)e^{inx}+(1-in)e^{-inx}=2(\cos nx-n\sin nx),\quad n=1,2,3......$

For $n=0$, $(-1)^n\left(\dfrac{1+in}{1+n^2}\right)e^{inx}=1$

Thus, $f(x)=\dfrac{\sinh\pi}{\pi}+2\dfrac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}(-1)^n\dfrac{(\cos nx-n\sin nx)}{1+n^2}$

$$f(x)=\frac{2\sinh\pi}{\pi}\left[\frac{1}{2}-\frac{1}{1+1^2}(\cos x-\sin x)+\frac{1}{1+2^2}(\cos 2x-2\sin 2x)-+...\right]$$

Example: Consider the periodic function $f(t)$ with time period T as shown in the figure below.



The spikes, located at $t=\frac{1}{2}(2n-1)$, where $n=0,\pm1,\pm2,...,$ are Dirac-delta function of strength ±1. Find the amplitudes a_n in the Fourier expansion of $f(t)=\sum_{n=-\infty}^{\infty} a_n e^{2\pi int/T}$.

Solution: $f(x)=\sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$, where $c_n=\frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}}dx$; $n=0,\pm1,\pm2.......$

and Range: $[-1,1]$, hence $2L=2$.

Comparing with $f(t)=\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n t/T}$, $L=\frac{T}{2}=1\Rightarrow T=2$.

$\Rightarrow f(t)=\sum_{n=-\infty}^{\infty} a_n e^{i\pi n t} \Rightarrow a_n=\frac{1}{2L}\int_{-L}^{L} f(t)e^{-i\pi n t}dt$

$\Rightarrow a_n=\frac{1}{2}\int_{-1}^{1} f(t)e^{-i\pi n t}dt=\frac{1}{2}\int_{-1}^{1}\left[\delta\left(t+\frac{1}{2}\right)-\delta\left(t-\frac{1}{2}\right)\right]e^{-i\pi n t}dt$

$\Rightarrow a_n=\frac{1}{2}\left[e^{-i\pi n(-1/2)}-e^{-i\pi n(1/2)}\right]=\frac{1}{2}\left[e^{i\pi n/2}-e^{-i\pi n/2}\right]=i\sin\frac{n\pi}{2}$

5.6 Approximation by Trigonometric Polynomials

Fourier series have major applications in approximation theory, that is, the approximation of functions by simpler functions.

Let $f(x)$ be a periodic function, of period 2π for simplicity that can be represented by a Fourier series. Then the N^{th} partial sum of the series is an approximation to $f(x)$

$$f(x)\approx a_0+\sum_{n=1}^{N}\left(a_n\cos nx+b_n\sin nx\right) \qquad ....(1)$$

We have to see whether (1) is the "best" approximation to f by a trigonometric polynomial of degree N, that is, by a function of the form

$$F(x)=A_0+\sum_{n=1}^{N}\left(A_n\cos nx+B_n\sin nx\right) \qquad ....(2)$$

where "best" means that the "error" of approximation is minimum.

The **total square error** of F relative to f on the interval $-\pi\le x\le\pi$ is given by

$$E=\int_{-\pi}^{\pi}(f-F)^2dx \qquad \text{clearly } E\ge 0.$$

The function F is a good approximation to f but $|f-F|$ is large at a point of discontinuity x_0.

5.6.1 Minimum square error

The total square error of F relative to f on the interval $-\pi\le x\le\pi$ is minimum if and only if the coefficients of $F(x)$ are the Fourier coefficients of $f(x)$. This minimum value E^* is given by

$$E^*=\int_{-\pi}^{\pi} f^2dx-\pi\left[2a_0^2+\sum_{n=1}^{N}\left(a_n^2+b_n^2\right)\right] \qquad ....(3)$$

From (3) we can see that E^* cannot increase as N increases, but may decrease. Hence with increasing N the partial sums of the Fourier series of f yields better and better approximations to f.

5.6.2 Parseval's Identity: Since $E^*\ge 0$ and equation (3) holds for every N, we obtain

$$2a_0^2+\sum_{n=1}^{N}\left(a_n^2+b_n^2\right)\le\frac{1}{\pi}\int_{-\pi}^{\pi} f^2dx$$

Now Parseval's Identity is $2a_0^2+\sum_{n=1}^{N}\left(a_n^2+b_n^2\right)=\frac{1}{\pi}\int_{-\pi}^{\pi} f^2dx$

Example: Compute the total square error of F with $N=3$ relative to

$$f(x)=x+\pi \qquad (-\pi<x<\pi)$$

on the interval $-\pi\le x\le\pi$.

Solution: Fourier coefficients are $a_0=\pi$, $a_n=0$ and $b_n=-\frac{2}{n}\cos n\pi$.

Its Fourier series is given by

$$F(x)=\pi+2\sin x-\sin 2x+\frac{2}{3}\sin 3x.$$

Hence,

$$E^*=\int_{-\pi}^{\pi} f^2dx-\pi\left[2a_0^2+\sum_{n=1}^{N}\left(a_n^2+b_n^2\right)\right]=\int_{-\pi}^{\pi}(x+\pi)^2dx-\pi\left[2\pi^2+2^2+1^2+\left(\frac{2}{3}\right)^2\right]$$

$$E^*=\frac{8}{3}\pi^3-\pi\left[2\pi^2+\frac{49}{9}\right]\approx 3.567$$

Although $|f(x)-F(x)|$ is large at $x=\pm\pi$, where f is discontinuous, F approximates f quite well on the whole interval.