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An Institute of NET-JRF, IIT-JAM, GATE, JEST,  
TIFR & CUET in Physics & Physical Sciences

Physics by fiziks

**Electromagnetic Theory**

(NET/JRF, GATE, JEST, TIFR)

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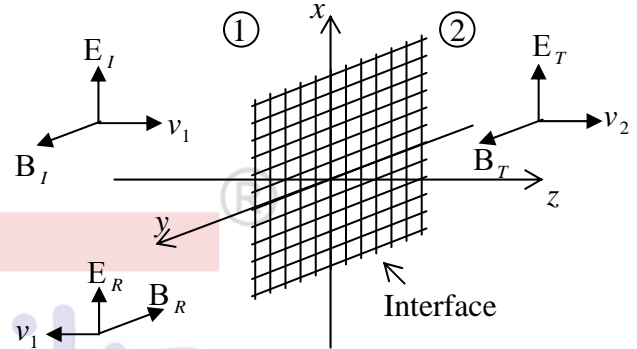
## CHAPTER-7

### APPLICATIONS OF ELECTROMAGNETIC WAVES

#### 7.1 Reflection and Refraction at Dielectric Interface

##### 7.1.1 Normal Incidence

Suppose  $xy$  plane forms the boundary between two linear media. A plane wave of frequency  $\omega$ , traveling in the  $z$ -direction and polarized in the  $x$ -direction, approaches the interface from the left then



##### Incident Wave

$$\left. \begin{aligned} \vec{E}_I(z,t) &= \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{x} \\ \vec{B}_I(z,t) &= \frac{\tilde{E}_{0I}}{v_1} e^{i(k_1 z - \omega t)} \hat{y} \end{aligned} \right\}$$

##### Reflected Wave

$$\left. \begin{aligned} \vec{E}_R(z,t) &= \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{x} \\ \vec{B}_R(z,t) &= -\frac{\tilde{E}_{0R}}{v_1} e^{i(-k_1 z - \omega t)} \hat{y} \end{aligned} \right\}$$

##### Transmitted Wave

$$\left. \begin{aligned} \vec{E}_T(z,t) &= \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{x} \\ \vec{B}_T(z,t) &= \frac{\tilde{E}_{0T}}{v_2} e^{i(k_2 z - \omega t)} \hat{y} \end{aligned} \right\}$$

At  $z=0$ , the combined field on the left  $\vec{E}_I + \vec{E}_R$  and  $\vec{B}_I + \vec{B}_R$ , must join the fields on the right  $\vec{E}_T$  &  $\vec{B}_T$ , in accordance with the **boundary conditions**

$$(i) \epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp \quad (ii) B_1^\perp = B_2^\perp \quad (iii) \vec{E}_1^\parallel = \vec{E}_2^\parallel \quad (iv) \frac{1}{\mu_1} \vec{B}_1^\parallel = \frac{1}{\mu_2} \vec{B}_2^\parallel$$

In this case, there are no electric component perpendicular to the surface, so (i) & (ii) are trivial. However (iii) gives  $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$

While (iv) gives,  $\frac{\tilde{E}_{0I}}{\mu_1 v_1} + \frac{(-\tilde{E}_{0R})}{\mu_1 v_1} = \frac{\tilde{E}_{0T}}{\mu_2 v_2}$  or  $\tilde{E}_{0I} - \tilde{E}_{0R} = \beta \tilde{E}_{0T}$  where  $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$ .

Solving above two equations we get  $\tilde{E}_{0R} = \left( \frac{1-\beta}{1+\beta} \right) \tilde{E}_{0I}$ ,  $\tilde{E}_{0T} = \left( \frac{2}{1+\beta} \right) \tilde{E}_{0I}$ .

If  $\mu_1 = \mu_2 = \mu_0 \Rightarrow \beta = \frac{v_1}{v_2} = \frac{n_2}{n_1}$  (For non-magnetic medium)

$$\Rightarrow \tilde{E}_{0R} = \left( \frac{v_2 - v_1}{v_2 + v_1} \right) \tilde{E}_{0I} \quad , \quad \tilde{E}_{0T} = \left( \frac{2v_2}{v_1 + v_2} \right) \tilde{E}_{0I}$$

**Note:** Reflected wave is in phase if  $v_2 > v_1$  or  $n_2 < n_1$  and out of phase if  $v_2 < v_1$  or  $n_2 > n_1$ .

In terms of indices of refraction the real amplitudes are

$$E_{0R} = \left| \frac{n_1 - n_2}{n_1 + n_2} \right| E_{0I} \quad , \quad E_{0T} = \left| \frac{2n_1}{n_1 + n_2} \right| E_{0I}$$

Since Intensity  $I = \frac{1}{2} \epsilon v E_0^2$ , then the ratio of the reflected intensity to the incident intensity is the

$$\text{Reflection coefficient } R = \frac{I_R}{I_I} = \left( \frac{E_{0R}}{E_{0I}} \right)^2 = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2.$$

The ratio of the transmitted intensity to the incident intensity is the *Transmission coefficient*

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \left( \frac{E_{0T}}{E_{0I}} \right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2} \Rightarrow R + T = 1$$

**Example:** Calculate the reflection coefficient for light at an air-to-dielectric interface ( $\mu_1 = \mu_2 = \mu_0$ ,  $n_1 = 1$ ,  $n_2 = 1.5$ ) at optical frequency  $\omega = 4 \times 10^{15} \text{ s}^{-1}$ .

**Solution:** Reflection coefficient  $R = \left( \frac{n_1 - n_2}{n_1 + n_2} \right)^2 = \left( \frac{1 - 1.5}{1 + 1.5} \right)^2 = 0.04$  or 4%

Thus only 4% of light is reflected and 96% is transmitted.

### 7.1.2 Oblique Incidence

In oblique incidence an incoming wave meets the boundary at an arbitrary angle  $\theta_i$ . Of course, normal incidence is really just a special case of oblique incidence with  $\theta_i = 0$ . Suppose that a monochromatic plane wave of frequency  $\omega$ , approaches the interface from the left then

Incident Wave

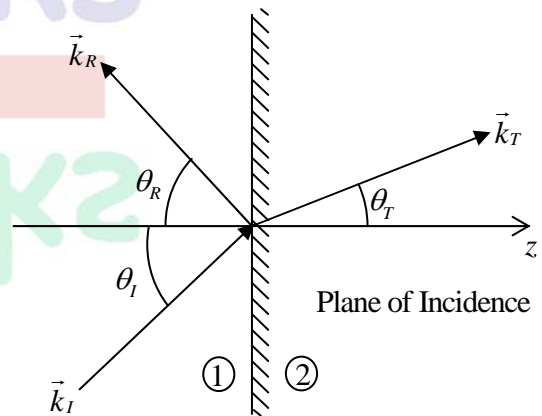
$$\vec{E}_I(\vec{r}, t) = \vec{E}_{0I} e^{i(\vec{k}_I \cdot \vec{r} - \omega t)}, \quad \vec{B}_I(\vec{r}, t) = \frac{1}{v_1} (\hat{k}_I \times \vec{E}_I)$$

Reflected Wave

$$\vec{E}_R(\vec{r}, t) = \vec{E}_{0R} e^{i(\vec{k}_R \cdot \vec{r} - \omega t)}, \quad \vec{B}_R(\vec{r}, t) = \frac{1}{v_1} (\hat{k}_R \times \vec{E}_R)$$

Transmitted Wave

$$\vec{E}_T(\vec{r}, t) = \vec{E}_{0T} e^{i(\vec{k}_T \cdot \vec{r} - \omega t)}, \quad \vec{B}_T(\vec{r}, t) = \frac{1}{v_2} (\hat{k}_T \times \vec{E}_T)$$



All three waves have the same frequency  $\omega$ . The three wave numbers are related by ( $\omega = kv$ ) as

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega \quad \text{or} \quad k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T$$

The combined field in medium (1),  $\vec{E}_I + \vec{E}_R$  and  $\vec{B}_I + \vec{B}_R$ , must join the fields  $\vec{E}_T$  &  $\vec{B}_T$  in medium (2), using the **boundary conditions**

$$(i) \quad \epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp \quad (ii) \quad B_1^\perp = B_2^\perp \quad (iii) \quad \vec{E}_1^\parallel = \vec{E}_2^\parallel \quad (iv) \quad \frac{1}{\mu_1} \vec{B}_1^\parallel = \frac{1}{\mu_2} \vec{B}_2^\parallel$$

**First Law (Plane of Incidence)**

The incident, reflected and transmitted wave vectors form a plane (called the plane of incidence), which also includes normal to the surface.

**Second law (Law of Reflection)**

The angle of incidence is equal to the angle of reflection i.e.

$$\theta_i = \theta_R$$

**Third Law: (Law of Refraction, or Snell's law)**

$$\frac{\sin \theta_T}{\sin \theta_I} = \frac{n_1}{n_2}$$

**7.1.3 Fresnel's Relation (Parallel and Perpendicular Polarization)**

**Case-I: (Polarization in the Plane of Incidence)**

Applying Boundary conditions, we get

Reflected and transmitted amplitudes

$$\tilde{E}_{0R} = \left( \frac{\alpha - \beta}{\alpha + \beta} \right) \tilde{E}_{0I} \quad \text{and} \quad \tilde{E}_{0T} = \left( \frac{2}{\alpha + \beta} \right) \tilde{E}_{0I}$$

where  $\alpha = \frac{\cos \theta_T}{\cos \theta_I}$  and  $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$

These are known as **Fresnel's equations**.

Notice that transmitted wave is always in phase with the incident one; the reflected wave is either in phase, if  $\alpha > \beta$ , or  $180^\circ$  out phase if  $\alpha < \beta$ .

The amplitudes of the transmitted and reflected waves depend on the angle of incidence, because  $\alpha$  is a function of  $\theta_I$ :

$$\alpha = \frac{\sqrt{1 - \sin^2 \theta_T}}{\cos \theta_I} = \frac{\sqrt{1 - \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta_I}}{\cos \theta_I}$$

**Brewster's Angle**

At *Brewster's angle* ( $\theta_B$ ) reflected light is completely extinguished when  $\alpha = \beta$ , or

$$\sin^2 \theta_B = \frac{1 - \beta^2}{\left( \frac{n_1}{n_2} \right)^2 - \beta^2}$$

For non-magnetic medium ( $\mu_1 \cong \mu_2$ ), so  $\beta \cong \frac{n_2}{n_1}$ ,  $\sin^2 \theta_B \cong \frac{\beta^2}{1 + \beta^2}$ , and hence

$$\tan \theta_B \cong \frac{n_2}{n_1} \quad \text{and} \quad \theta_T + \theta_B = 90^\circ$$

Thus at Brewster angle ( $\theta_I = \theta_B$ ) reflected and transmitted rays are perpendicular to each other.

**Critical Angle**

When light enters from denser to rarer medium ( $n_1 > n_2$ ) then after a **critical angle** ( $\theta_c$ ) there is total internal reflection.

$$\frac{\sin 90^\circ}{\sin \theta_c} = \frac{n_1}{n_2} \Rightarrow \sin \theta_c = \frac{n_2}{n_1} \quad \text{at } \theta_c, \theta_T = 90^\circ$$

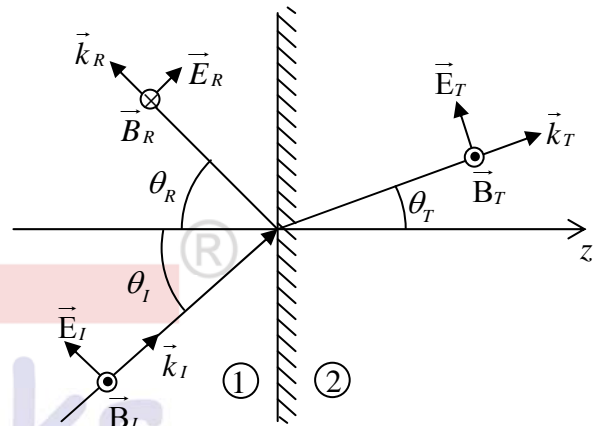
**Reflection and Transmission Coefficient**

The power per unit area striking the interface is  $\vec{S} \cdot \hat{z}$ . Thus the incident intensity is

$$I_I = \frac{1}{2} \epsilon_1 v_1 E_{0I}^2 \cos \theta_I,$$

while reflected and transmitted intensities are

$$I_R = \frac{1}{2} \epsilon_1 v_1 E_{0R}^2 \cos \theta_R \quad \text{and} \quad I_T = \frac{1}{2} \epsilon_2 v_2 E_{0T}^2 \cos \theta_T$$



Reflection coefficient  $R = \frac{I_R}{I_I} = \left( \frac{E_{0R}}{E_{0I}} \right)^2 = \left( \frac{\alpha - \beta}{\alpha + \beta} \right)^2$

Transmission coefficient  $T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \left( \frac{E_{0T}}{E_{0I}} \right)^2 \frac{\cos \theta_T}{\cos \theta_I} = \alpha \beta \left( \frac{2}{\alpha + \beta} \right)^2$

$\Rightarrow R + T = 1$

**Case-II: (Polarization Perpendicular to plane of Incidence)**

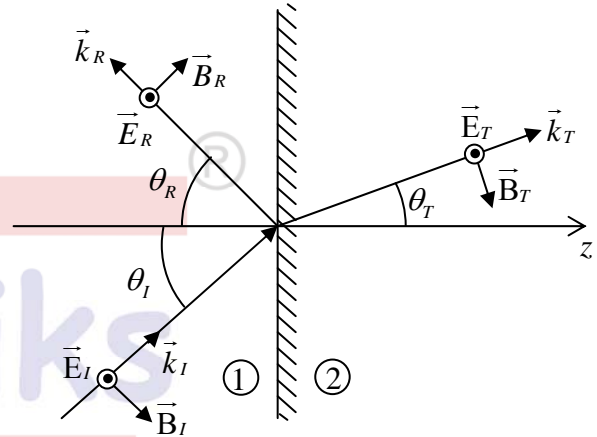
Applying Boundary conditions, we get

Reflected and transmitted amplitudes

$\hat{E}_{0R} = \left( \frac{1 - \alpha\beta}{1 + \alpha\beta} \right) \hat{E}_{0I}$  and  $\hat{E}_{0T} = \left( \frac{2}{1 + \alpha\beta} \right) \hat{E}_{0I}$

where  $\alpha = \frac{\cos \theta_T}{\cos \theta_I}$  and  $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$

In this case Brewster's angle ( $\theta_B$ ) is not possible i.e. reflected light is never completely extinguished (since  $\alpha\beta = 1$  is not possible).



**Reflection and Transmission coefficient**

The power per unit area striking the interface is  $\vec{S} \cdot \hat{z}$ . Thus the incident intensity is

$I_I = \frac{1}{2} \epsilon_1 v_1 E_{0I}^2 \cos \theta_I$ ,

while reflected and transmitted intensities are

$I_R = \frac{1}{2} \epsilon_1 v_1 E_{0R}^2 \cos \theta_R$  and  $I_T = \frac{1}{2} \epsilon_2 v_2 E_{0T}^2 \cos \theta_T$

Reflection coefficient  $R = \frac{I_R}{I_I} = \left( \frac{E_{0R}}{E_{0I}} \right)^2 = \left( \frac{1 - \alpha\beta}{1 + \alpha\beta} \right)^2$

Transmission coefficient  $T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \left( \frac{E_{0T}}{E_{0I}} \right)^2 \frac{\cos \theta_T}{\cos \theta_I} = \alpha \beta \left( \frac{2}{1 + \alpha\beta} \right)^2$

$\Rightarrow R + T = 1$

**Example:** A plane electromagnetic wave from within a dielectric medium (with  $\epsilon = 4\epsilon_0$  and  $\mu = \mu_0$ ) is incident on its boundary with air, at  $z = 0$ . The magnetic field in the medium is

$\vec{H} = \hat{j} H_0 \cos(\omega t - k\sqrt{3}x - kz)$ , where  $\omega$

and  $k$  are positive constants.

Find the angle of refraction.

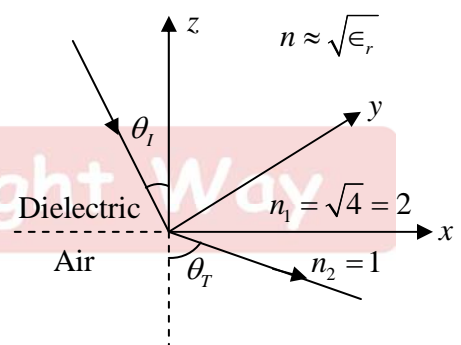
**Solution:**

Refractive index  $n \approx \sqrt{\epsilon_r}$  and  $\vec{k} = k\sqrt{3}\hat{x} + k\hat{z}$ .

$\frac{\sin \theta_I}{\sin \theta_T} = \frac{n_2}{n_1} = \frac{1}{2}$

$\sin \theta_T = 2 \sin \theta_I$ ;  $\therefore \tan \theta_I = \frac{k_x}{k_z} = \frac{\sqrt{3}}{1} \Rightarrow \theta_I = 60^\circ$

$\Rightarrow \sin \theta_T = 2 \times \sin 60^\circ = \sqrt{3} \Rightarrow \theta_T = \sin^{-1}(\sqrt{3})$



**Example:** A light source is at the bottom of a pool of water (index of refraction of water is 1.33). At what minimum angle of incidence will a ray be totally reflected at the surface?

**Solution:**

Total internal reflection is when one has a beam of light having all of the incident wave reflected.

Law for Total Internal Reflection,

$$n_{\text{inside}} \sin \theta = n_{\text{outside}},$$

where  $n_{\text{inside}} = 1.33$ , and one assumes that the surface has  $n_{\text{outside}} = 1$  for air.

$$\Rightarrow \theta = \sin^{-1}(1/1.33) = \sin^{-1}(0.76) = 50^\circ$$

**Example:** Unpolarized light is incident on an air-dielectric interface. The interface is the  $xy$  plane, and the plane of incidence is  $yz$  plane. The electric field of the reflected light is given

by  $\vec{E} = E_0 \hat{x} \exp\left\{\frac{ik}{2}(\sqrt{3}y + z) - i\omega t\right\}$ , where  $k$  is the propagation constant in air and  $\omega$  is the angular frequency of the light. Assume magnetic permeability  $\mu = \mu_0$ . Find the dielectric constant of the second medium.

**Solution:**

Since reflected light is plane polarized, then angle of reflection is Brewster angle

$$\tan \theta_R = \tan \theta_B = \left(\frac{n_2}{n_1}\right) = n_2 \Rightarrow \vec{k} = \frac{k}{2}(\sqrt{3}\hat{y} + \hat{z}) \Rightarrow \tan \theta_B = \left(\frac{k_y}{k_z}\right) = \sqrt{3} \Rightarrow n_2 = \sqrt{\epsilon_r} = \sqrt{3} \Rightarrow \epsilon_r = 3$$

## 7.2 Reflection at Conducting Surface (Normal Incidence)

Suppose  $xy$  plane forms the boundary between a non-conducting linear medium (1) and a conductor (2). A plane wave of frequency  $\omega$ , traveling in the  $z$ -direction and polarized in the  $x$  direction, approaches the interface from the left then

Incident Wave

$$\left. \begin{aligned} \vec{E}_I(z, t) &= \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{x} \\ \vec{B}_I(z, t) &= \frac{\tilde{E}_{0I}}{v_1} e^{i(k_1 z - \omega t)} \hat{y} \end{aligned} \right\}$$

Reflected Wave

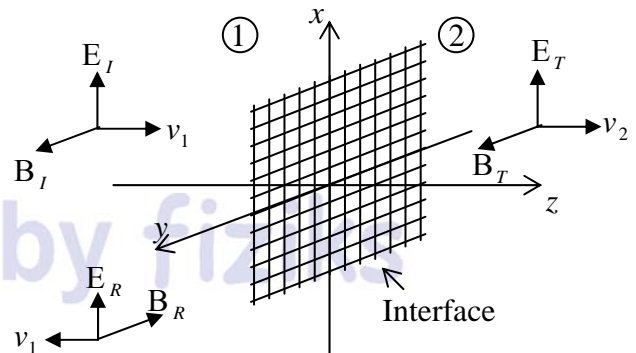
$$\left. \begin{aligned} \vec{E}_R(z, t) &= \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{x} \\ \vec{B}_R(z, t) &= -\frac{\tilde{E}_{0R}}{v_1} e^{i(-k_1 z - \omega t)} \hat{y} \end{aligned} \right\}$$

Transmitted Wave

$$\left. \begin{aligned} \vec{E}_T(z, t) &= \tilde{E}_{0T} e^{i(\tilde{k}_2 z - \omega t)} \hat{x} \\ \vec{B}_T(z, t) &= \frac{\tilde{k}_2}{\omega} \tilde{E}_{0T} e^{i(\tilde{k}_2 z - \omega t)} \hat{y} \end{aligned} \right\}$$

where  $\tilde{k}_2 = k_2 + i\kappa_2$  where  $k_2$  and  $\kappa_2$  are real and imaginary part of  $\tilde{k}_2$ .

$$k_2 = \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1 \right]^{1/2} \quad \text{and} \quad \kappa_2 = \omega \sqrt{\frac{\epsilon\mu}{2}} \left[ \sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1 \right]^{1/2}$$



At  $z=0$ , the combined field on the left  $\tilde{E}_I + \tilde{E}_R$  and  $\tilde{B}_I + \tilde{B}_R$ , must join the fields on the right  $\tilde{E}_T$  &  $\tilde{B}_T$ , in accordance with the **boundary conditions**

$$(i) \epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp \quad (ii) B_1^\perp = B_2^\perp \quad (iii) \vec{E}_1^\parallel = \vec{E}_2^\parallel \quad (iv) \frac{1}{\mu_1} \vec{B}_1^\parallel = \frac{1}{\mu_2} \vec{B}_2^\parallel$$

In this case there are no electric component perpendicular to the surface, so (i) & (ii) are trivial.

However (iii) gives  $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$

While (iv) gives, 
$$\frac{\tilde{E}_{0I}}{\mu_1 v_1} + \frac{(-\tilde{E}_{0R})}{\mu_1 v_1} = \frac{\tilde{k}_2}{\mu_2 \omega} \tilde{E}_{0T} \quad \text{or} \quad \tilde{E}_{0I} - \tilde{E}_{0R} = \tilde{\beta} \tilde{E}_{0T} \quad \text{where} \quad \tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}_2.$$

Solving above two equations we get 
$$\tilde{E}_{0R} = \left( \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \right) \tilde{E}_{0I}, \quad \tilde{E}_{0T} = \left( \frac{2}{1+\tilde{\beta}} \right) \tilde{E}_{0I}.$$

**Note:** (i) For a perfect conductor ( $\sigma = \infty$ ),  $k_2 = \infty \Rightarrow \tilde{\beta} = \infty$ . Thus

$$\tilde{E}_{0R} = -\tilde{E}_{0I}, \quad \tilde{E}_{0T} = 0.$$

In this case wave is totally reflected, with a  $180^\circ$  phase shift.

(ii) For good conductor ( $\sigma \gg \omega\epsilon$ ),  $k_2 \cong \kappa_2 = \sqrt{\frac{\sigma\omega\mu_2}{2}}$ .

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega} \sqrt{\frac{\sigma\omega\mu_2}{2}} (1+i) = \mu_1 v_1 \sqrt{\frac{\sigma}{2\omega\mu_2}} (1+i) \Rightarrow \tilde{\beta} = \gamma(1+i) \quad \text{where} \quad \gamma = \mu_1 v_1 \sqrt{\frac{\sigma}{2\omega\mu_2}}.$$

Reflection Coefficient: 
$$R = \frac{I_R}{I_I} = \left| \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} \right|^2 = \left| \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \right|^2 = \left( \frac{1-\gamma-i\gamma}{1+\gamma+i\gamma} \right) \left( \frac{1-\gamma+i\gamma}{1+\gamma-i\gamma} \right) = \frac{(1-\gamma)^2 + \gamma^2}{(1+\gamma)^2 + \gamma^2}$$

**Example:** Calculate the reflection coefficient for light at an air-to-silver interface

( $\mu_1 = \mu_2 = \mu_0$ ,  $\epsilon_1 = \epsilon_0$ ,  $\sigma = 6 \times 10^7 \Omega^{-1} m^{-1}$ ) at optical frequency  $\omega = 4 \times 10^{15} s^{-1}$ .

**Solution:** 
$$\gamma = \mu_0 c \sqrt{\frac{\sigma}{2\omega\mu_0}} = c \sqrt{\frac{\sigma\mu_0}{2\omega}} = (3 \times 10^8) \sqrt{\frac{(6 \times 10^7)(4\pi \times 10^{-7})}{2(4 \times 10^{15})}} = 29$$

Reflection coefficient

$$R = \frac{(1-\gamma)^2 + \gamma^2}{(1+\gamma)^2 + \gamma^2} = \frac{(28)^2 + 29^2}{(30)^2 + 29^2} = 0.93 \quad \text{or} \quad 93\%.$$

Thus 93% of light is reflected and only 7% is transmitted.

**Example:** A beam of light of frequency  $\omega$  is reflected from a dielectric-metal interface at normal incidence. The refractive index of the dielectric medium is  $n$  and that of the metal is  $n_2 = n(1+i\rho)$ . If the beam is polarised parallel to the interface, then find the phase change experienced by the light upon reflection.

**Solution:**

Since 
$$\tilde{E}_{0R} = \left( \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \right) \tilde{E}_{0I} \quad \text{where} \quad \tilde{\beta} = \frac{v_1}{v_2} = \frac{c/n}{c/n(1+i\rho)} = 1+i\rho$$

$$\Rightarrow \tilde{E}_{0R} = \left( \frac{-i\rho}{2+i\rho} \right) \tilde{E}_{0I} = \left( \frac{\rho e^{-i\pi/2}}{\sqrt{4+\rho^2} e^{i\theta}} \right) \tilde{E}_{0I} = \left( \frac{\rho}{\sqrt{4+\rho^2}} \right) e^{-i(\pi/2+\theta)} \tilde{E}_{0I} \quad \text{where} \quad \tan \theta = \frac{\rho}{2}.$$

Thus phase change 
$$\phi = -(\pi/2 + \theta) \Rightarrow \tan \phi = \cot \theta = \frac{2}{\rho} \Rightarrow \phi = \tan^{-1} \left( \frac{2}{\rho} \right)$$