CHAPTER 5
FOURIER SERIES

5.1 Periodic Functions and Trigonometric Series

5.1.1 Periodic Functions

A function $f(x)$ is called periodic if it is defined for all (except for certain isolated x such as $\pm\pi/2$, $\pm 3\pi/2$……, for $\tan x$ whose period is π) real x and if there is some positive number p such that

$$f(x+p) = f(x) \qquad \text{for all } x.$$

The number p is called **period** of $f(x)$. The graph of such function is obtained by periodic repetition of its graph in any interval of length p.

Fundamental Period: If a periodic function $f(x)$ has a smallest period $p\,(>0)$, this is often called the fundamental period of $f(x)$.

NOTE: (i) Familiar periodic functions are sine and cosine functions.

(ii) The function $f = \text{constant}$ is also a periodic function.

(iii) The functions that are not periodic are x, x^2, x^3, e^x, $\cosh x$, $\ln x$ etc.

(iv) $\because f(x+2p) = f[(x+p)+p] = f(x+p) = f(x)$.

Thus, for any integer n, $f(x+np) = f(x)$. Hence $2p, 3p, \ldots$ are also period of $f(x)$.

(v) If $f(x)$ and $g(x)$ have period p, then the function $h(x) = af(x) + bg(x)$ (a, b constants) has also period p.

Example: (i) For $\sin x$ and $\cos x$ the fundamental period is 2π.

(ii) For $\sin 2x$ and $\cos 2x$ the fundamental period is π.

(iii) For $\tan x$ and $\cot x$ the fundamental period is π.

(iv) For $\sin \pi x$ and $\cos \pi x$ the fundamental period is 2.

(v) For $\sin 2\pi x$ and $\cos 2\pi x$ the fundamental period is 2.

(vi) A function without fundamental period is $f = \text{constant}$.

5.1.2 Trigonometric Series

Let's represent various functions of period $p = 2\pi$ in terms of simple functions

$$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x, \ldots\ldots\ \cos nx,\ \sin nx, \ldots\ldots$$

These functions have period 2π. Figure below shows the first few of them.

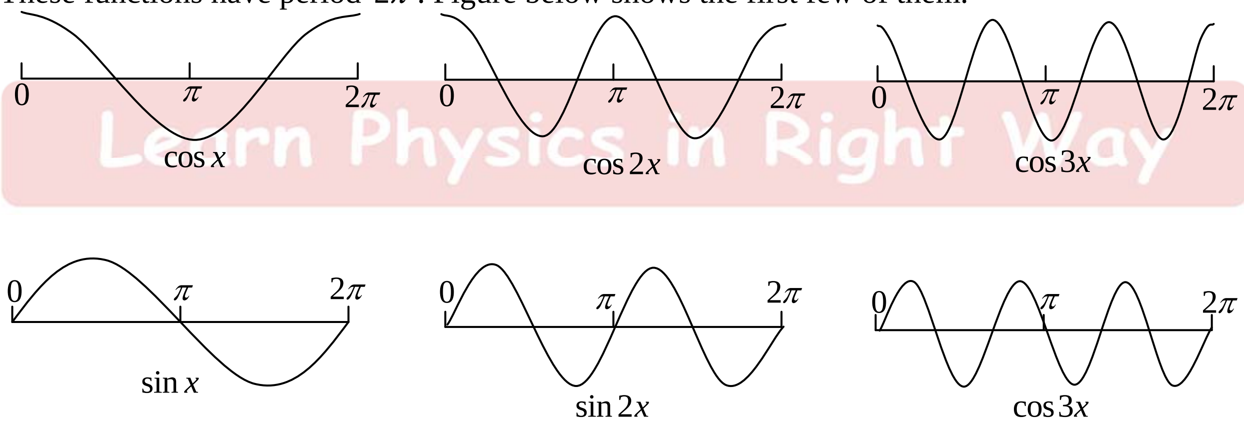


Figure: Cosine and sine functions having the period 2π

The series that will arise in this connection will be of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + ...$$

where $a_0, a_1, a_2 ... b_1, b_2 ...$ are real constants. Such a series is called trigonometric series and the a_n and b_n are called the coefficient of the series. Thus, we may write series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

We see that each term of the series has the period 2π. Hence if the series converges, its sum will be a function of period 2π.

NOTE:

The trigonometric series can be used for representing any practically important periodic function f, simple or complicated, of any period p. This series will then be called the Fourier series of f.

5.2 Fourier Series

Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine and sine functions. These series are trigonometric series whose coefficients are determined from $f(x)$ by the "Euler Formulas".

Condition for Existence of Fourier Series

(i) $f(x)$ is periodic and single valued.

(ii) $f(x)$ is finite at all points in the given interval (i.e. it is bounded and have upper limit).

(iii) $f(x)$ may have finite number of discontinuities (i.e. it may be piece-wise continuous).

(iv) $f(x)$ may have finite number of maxima or minima or both.

Let us assume that $f(x)$ is periodic function of period 2π and is integrable over a period. Let us further assume that $f(x)$ can be represented by a trigonometric series, (assume that this series converges and has $f(x)$ as its sum)

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad ....(1)$$

5.2.1 Euler Formulas for the Fourier Coefficients

Determination of constant a_0:

From equation (1), we have

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right)$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0 \qquad \because \int_{-\pi}^{\pi} \cos nx dx = 0, \quad \int_{-\pi}^{\pi} \sin nx dx = 0$$

Thus,

$$\boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx} \quad ....(2)$$

H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

Determination of constant a_n :

Multiply equation (1) by $\cos mx$, where m is any fixed positive integer, then

$$\int_{-\pi}^{\pi} f(x)\cos mx dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)\right]\cos mx dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos mx dx = a_0\int_{-\pi}^{\pi}\cos mx dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos mx dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx dx\right)$$

$$\because \int_{-\pi}^{\pi}\cos mx dx = 0$$

$$\int_{-\pi}^{\pi}\cos nx\cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)x dx = \begin{cases}0, & m\neq n\\ \pi, & m=n\end{cases}$$

$$\int_{-\pi}^{\pi}\sin nx\cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi}\sin(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi}\sin(n-m)x dx = 0, \text{ always}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos nx dx = a_n\pi, \text{ when } n=m$$

Thus,

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx} \quad \text{....(3)}$$

Determination of constant b_n :

Multiply equation (1) by $\sin mx$, where m is any fixed positive integer, then

$$\int_{-\pi}^{\pi} f(x)\sin mx dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)\right]\sin mx dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\sin mx dx = a_0\int_{-\pi}^{\pi}\sin mx dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\sin mx dx + b_n\int_{-\pi}^{\pi}\sin nx\sin mx dx\right)$$

$$\int_{-\pi}^{\pi}\cos nx\sin mx dx = \frac{1}{2}\int_{-\pi}^{\pi}\sin(n+m)x dx + \frac{1}{2}\int_{-\pi}^{\pi}\sin(m-n)x dx = 0, \text{ always., } \quad \int_{-\pi}^{\pi}\sin mx dx = 0$$

$$\int_{-\pi}^{\pi}\sin nx\sin mx dx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)x dx - \frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)x dx = \begin{cases}0, & m\neq n\\ \pi, & m=n\end{cases}.$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\sin nx dx = b_n\pi \text{ when } n=m;$$

Thus,

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx} \quad \text{....(4)}$$

5.2.2 Orthogonality of the Trigonometric System

The trigonometric system

$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x, \ldots\ldots \cos nx,\ \sin nx, \ldots\ldots$

is orthogonal on the interval $-\pi \le x \le \pi$ (hence on any interval of length 2π, because of periodicity). Thus, for any integer m and n we have

$$\int_{-\pi}^{\pi}\cos mx\cos nx dx = 0 \qquad (m\neq n)$$

and

$$\int_{-\pi}^{\pi}\sin mx\sin nx dx = 0 \qquad (m\neq n)$$

and for any integer m and n (including $m=n$) we have $\int_{-\pi}^{\pi}\cos mx\sin nx dx = 0$

5.2.3 Convergence and Sum of Fourier Series

If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \le x \le \pi$ and has left hand and right hand derivative at each point of that interval, then the Fourier series of $f(x)$ with coefficient a_0, a_n, b_n is convergent.

Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left and right-hand limit of $f(x)$ at x_0.

NOTE: (i) The left-hand limit of $f(x)$ at x_0 is $f(x_0 - 0) = \lim_{h \to 0} f(x_0 - h)$.

The right-hand limit of $f(x)$ at x_0 is $f(x_0 + 0) = \lim_{h \to 0} f(x_0 + h)$.

(ii) Function $f(x)$ is continuous at x_0, if

$$f(x_0 - 0) = f(x_0 + 0) = f(x_0)$$

(iii) The left-hand derivative of $f(x)$ at x_0 is $\lim_{h \to 0} \frac{f(x_0 - h) - f(x_0)}{-h}$.

The right-hand derivative of $f(x)$ at x_0 is $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

Function $f(x)$ is differentiable at x_0, if $L.H.D. = R.H.D.$

Some Basic Mathematics Result to be use in Fourier Series

(i) $\int u(x) v(x) dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$

where $u' = \frac{du}{dx}, u'' = \frac{d^2u}{dx^2}, ...$ and $v_1 = \int v dx, v_2 = \int v_1 dx, ...$

Example: $\int x^3 \sin nx dx = x^3 \left(-\frac{\cos nx}{n}\right) - 3x^2 \left(-\frac{\sin nx}{n^2}\right) + 6x \left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)$

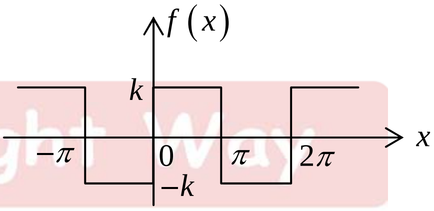
(i) $\cos n\pi = (-1)^n,\ n = 0, 1, 2...,\quad \sin n\pi = 0,\ n = 0, 1, 2...$

$$\cos n\frac{\pi}{2} = \begin{cases} 0, & n = 1, 3, 5, ... \\ 1, & n = 0, 4, 8... \\ -1, & n = 2, 6, 10... \end{cases}, \quad \sin n\frac{\pi}{2} = \begin{cases} 0, & n = 0, 2, 4... \\ 1, & n = 1, 5, 9... \\ -1, & n = 3, 7, 11... \end{cases}$$

Example: Find the Fourier coefficient of the periodic function $f(x)$ as shown in figure:

$$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x)$$

Hence show that: $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$



Solution: Let $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$\because a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$\Rightarrow a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \left[\int_{-\pi}^{0} (-k) dx + \int_{0}^{\pi} (k) dx \right] = \frac{1}{2\pi} \left[[-kx]_{-\pi}^{0} + [kx]_{0}^{\pi} \right] = 0$$

This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ to π is zero.

H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

$$\because a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx dx + \int_{0}^{\pi}(k)\cos nx dx\right] = \frac{1}{\pi}k\left[-\left\{\frac{\sin nx}{n}\right\}_{-\pi}^{0} + \left\{\frac{\sin nx}{n}\right\}_{0}^{\pi}\right] = 0$$

$$\because b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx dx + \int_{0}^{\pi}(k)\sin nx dx\right]$$

$$\Rightarrow b_n = \frac{1}{\pi}k\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{0} - \left\{\frac{\cos nx}{n}\right\}_{0}^{\pi}\right] = \frac{1}{\pi}k\left[\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n}\right] = \frac{1}{\pi}k\left[\frac{2}{n} - \frac{2(-1)^n}{n}\right]$$

If n is even $b_n = 0$ and if n is odd $b_n = \frac{4k}{n\pi}$.

Thus, Fourier series is $f(x) = \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]$

The partial sums are $S_1 = \frac{4k}{\pi}\sin x$, $S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$, etc

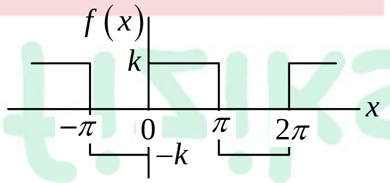


Figure (a): The given function $f(x)$ (Period square wave)

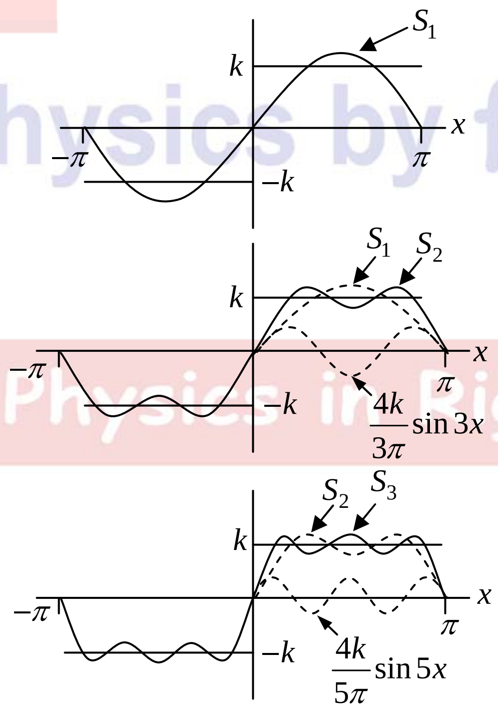


Figure (b): The first three partial sums of the corresponding Fourier series

NOTE: The above graph seems to indicate that the series is convergent and has the sum $f(x)$, the given function. Notice that at $x=0$ and $x=\pi$, the points of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the values $-k$ and $+k$ of our function.

Assuming that $f(x)$ is the sum of the series and setting $x=\frac{\pi}{2}$, we have

$$f\left(\frac{\pi}{2}\right)=k=\frac{4k}{\pi}\left[\sin\frac{\pi}{2}+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\sin\frac{5\pi}{2}+...\right]$$

$$\Rightarrow 1=\frac{4}{\pi}\left[1+\frac{1}{3}(-1)+\frac{1}{5}(1)+\frac{1}{7}(-1)+...\right]=\frac{4}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...\right] \Rightarrow \frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...$$

Example: The square wave in previous example has a jump at $x=0$. Its left-hand limit there is $-k$ and its right-hand limit there is $+k$. Hence average of these limits is 0. Thus, Fourier series converge to this value at $x=0$, because then all its terms are 0. Similarly, for the other jump we can verify this.

Example: Find the Fourier coefficient of the periodic function $f(x)$:

$$f(x)=x, \quad -\pi<x<\pi \qquad \text{having period } 2\pi$$

Solution: $\because a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \Rightarrow a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} xdx=0$

$$\because a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx \Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nxdx=\frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right)-1\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}=0$$

$$\because b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx \Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nxdx=\frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-1\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$\Rightarrow b_n=\frac{1}{\pi n}\left[-\pi\cos n\pi-\pi\cos n\pi\right]=-\frac{2}{n}(-1)^n$$

Thus $f(x)=-2\left[-\frac{\sin x}{1}+\frac{\sin 2x}{2}-\frac{\sin 3x}{3}+...\right]=2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-...\right]$

Example: Find the Fourier coefficient of the periodic function $f(x)$:

$$f(x)=x^2, \quad -\pi<x<\pi \qquad \text{having period } 2\pi$$

Solution: $\because a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \Rightarrow a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2dx=\frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}=\frac{1}{6\pi}\left[\pi^3+\pi^3\right]=\frac{\pi^2}{3}$

$$\because a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

$$\Rightarrow a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nxdx=\frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-2x\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$\Rightarrow a_n=\frac{2}{n^2\pi}\left[\pi\cos n\pi+\pi\cos n\pi\right]=\frac{4}{n^2}(-1)^n$$

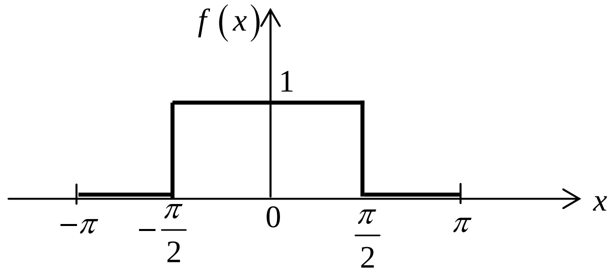
$$\because b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$$

$$\Rightarrow b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nxdx=\frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right)-2x\left(-\frac{\sin nx}{n^2}\right)+\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}=0$$

Thus $f(x)=\frac{\pi^2}{3}-4\left(\cos x-\frac{1}{4}\cos 2x+\frac{1}{9}\cos 3x-....\right)$

H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

Example: Find the Fourier series of the periodic function $f(x)$ as shown in figure:



Solution: The given function is periodic in 2π .

$$\because a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \Rightarrow a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2}(0)\,dx + \int_{-\pi/2}^{\pi/2}(1)\,dx\right] = \frac{1}{2\pi}\times\pi = \frac{1}{2}$$

$$\because a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(0)\cos nx dx + \int_{-\pi/2}^{\pi/2}(1)\cos nx dx\right] = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi/2}^{\pi/2} = \frac{2}{n\pi}\sin n\frac{\pi}{2}$$

$$\because b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \Rightarrow b_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(0)\sin nx dx + \int_{-\pi/2}^{\pi/2}(1)\sin nx dx\right] = \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\pi/2}^{\pi/2} = 0$$

$$\Rightarrow b_n = \frac{1}{\pi}k\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{0} - \left\{\frac{\cos nx}{n}\right\}_{0}^{\pi}\right] = \frac{1}{\pi}k\left[\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n}\right] = \frac{1}{\pi}k\left[\frac{2}{n} - \frac{2(-1)^n}{n}\right]$$

$$\because \sin n\frac{\pi}{2} = \begin{cases} 0, & n = 2,4,6... \\ 1, & n = 1,5,9... \\ -1, & n = 3,7,11... \end{cases}$$

Thus, Fourier series is $f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + ...\right]$

Example: Find the Fourier coefficient of the periodic function $f(x)$:

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 0\ , & 0 < x < \pi \end{cases} \quad \text{having period } 2\pi$$

Solution: $\because a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \Rightarrow a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(1)\,dx + \int_{0}^{\pi}(0)\,dx\right] = \frac{1}{2\pi}\times\pi = \frac{1}{2}$

$$\because a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(1)\cos nx dx + \int_{0}^{\pi}(0)\cos nx dx\right] = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} = 0$$

$$\because b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \Rightarrow b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(1)\sin nx dx + \int_{0}^{\pi}(0)\sin nx dx\right] = \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0}$$

$$\Rightarrow b_n = -\frac{1}{n\pi}[1 - \cos n\pi] = -\frac{1}{n\pi}\left[1 - (-1)^n\right] = -\frac{2}{n\pi},\ n = 1,3,..\ \text{and}\ \ 0,\ n = 2,4,..$$

Thus, Fourier series is

$$f(x) = \frac{1}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right]$$

Example: Find the Fourier coefficient of the periodic function $f(x)$:

$$f(x) = x, \quad 0 < x < 2\pi \qquad \text{having period } 2\pi$$

Solution: $\because a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} x dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi}\times\frac{4\pi^2}{2} = \pi$

$$\because a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx \Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx dx = \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = 0$$

$$\because b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx \Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx dx = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$\Rightarrow b_n = \frac{1}{\pi n}\left[-2\pi\cos 2n\pi + 0\right] = -\frac{2}{n}$$

$$\because f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx), \quad f(x) = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x...\right]$$

Example: Find the Fourier coefficient of the periodic function $f(x)$:

$$f(x) = x^2, \quad 0 < x < 2\pi \qquad \text{having period } 2\pi$$

Solution: $\because a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{6\pi}\times 8\pi^3 = \frac{4}{3}\pi^2$

$$\because a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx dx = \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$\Rightarrow a_n = \frac{2}{n^2\pi}\left[2\pi\cos 2n\pi + 0\right] = \frac{2}{n^2\pi}\times 2\pi = \frac{4}{n^2}$$

$$\because b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx dx = \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + \left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi} = -\frac{4\pi}{n}$$

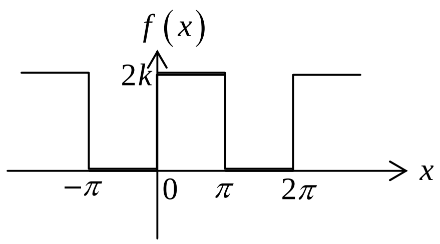
$$f(x) = \frac{4}{3}\pi^2 + 4\left(\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - ....\right) - 4\pi\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...\right)$$

Sum and Scalar Multiple

(a) The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

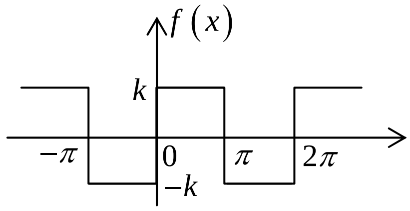
(b) The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

Example: Find the Fourier series of the periodic function $f(x)$ as shown in figure:



Solution: We have already calculated the Fourier series of the periodic function $f(x)$ as shown in figure.

The Fourier series is $f(x)=\frac{4k}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...\right]$.



The function given in the problem can be obtained by adding k to the above function. Thus, the Fourier series of a sum $k+f(x)$ are the sums of the corresponding Fourier series of k and $f(x)$.

The Fourier series is $f(x)=k+\frac{4k}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...\right]$.

Example: Find the Fourier series of the periodic function

$$f(x)=x+\pi\ ;\quad(-\pi<x<\pi)\ \text{ having period } 2\pi$$

Solution: Let $f(x)=f_1+f_2$, where $f_1=x$ and $f_2=\pi$.

The Fourier coefficient of $f_2=\pi$ is $a_0=\pi$, $a_n=0$ and $b_n=0$.

The Fourier coefficient of $f_1=x$ is $a_0=0$, $a_n=0$ and $b_n=-2\frac{(-1)^n}{n}$; $n=1,2,3....$.

Thus, the Fourier series of $f(x)$ is $f(x)=\pi+2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x...\right]$.

Example: Find the Fourier series of the periodic function

$$f(x)=x+x^2\ ;\quad(-\pi<x<\pi)\ \text{ having period } 2\pi$$

Solution: Let $f(x)=f_1+f_2$, where $f_1=x$ and $f_2=x^2$.

Thus $f_1(x)=2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-...\right]$, $f_2(x)=\frac{\pi^2}{3}-4\left(\cos x-\frac{1}{4}\cos 2x+\frac{1}{9}\cos 3x-....\right)$

Thus, the Fourier series of $f(x)$ is

$$f(x)=\frac{\pi^2}{3}-4\left(\cos x-\frac{1}{4}\cos 2x+\frac{1}{9}\cos 3x-....\right)+2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-...\right].$$

Example: Find the Fourier series of the periodic function

$$f(x)=x+x^2\ ;\quad(0<x<2\pi)\ \text{ having period } 2\pi$$

Solution: Let $f(x)=f_1+f_2$, where $f_1=x$ and $f_2=x^2$.

Thus $f_1(x)=\pi-2\left[\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\frac{1}{4}\sin 4x...\right]$,

$$f_2(x)=\frac{4}{3}\pi^2+4\left(\cos x+\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x-....\right)-4\pi\left(\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+....\right)$$

Thus, the Fourier series of $f(x)$ is $f(x)=f_1(x)+f_2(x)$

5.3 Function of Any Period $p = 2L$

The functions considered so far had period 2π, for simplicity. The transition from $p = 2\pi$ to $p = 2L$ is quite simple. It amounts to a stretch (or contraction) of scale on the axis.

Fourier series of a function $f(x)$ of period $p = 2L$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right) \qquad \text{....(1)}$$

where Fourier coefficients of $f(x)$ are

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

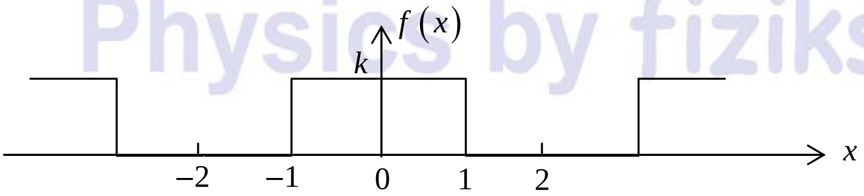
Let $v = \frac{\pi x}{L} \Rightarrow x = \frac{Lv}{\pi}$ and $dv = \frac{\pi dx}{L}$. Also $x = \pm L$ corresponds to $v = \pm\pi$.

Thus $f(x) = g(v)$ has period 2π.

Hence, we can verify that $g(v) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)\,dv$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\,dv$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\sin nv\,dv$

Example: Find the Fourier series of the function $f(x)$ as shown in figure:



$$f(x) = \begin{cases} 0 & \text{if} \quad -2 < x < -1 \\ k & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < 2 \end{cases} \qquad p = 2L = 4,\ L = 2$$

Solution: Let $f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$

$$\because a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$\Rightarrow a_0 = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} k\,dx = \frac{k}{2}$$

$$\because a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$\Rightarrow a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}dx$$

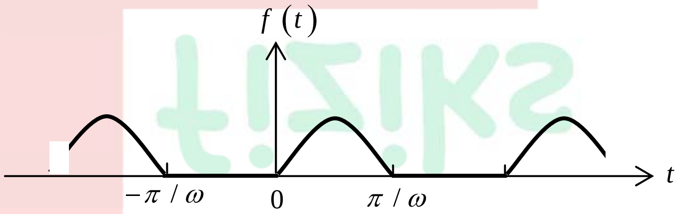
$$\Rightarrow a_n = \frac{k}{2}\left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_{-1}^{1} = \frac{2k}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 0, & n = 2,4,6..... \\ \dfrac{2k}{n\pi}, & n = 1,5,9..... \\ -\dfrac{2k}{n\pi}, & n = 3,7,11..... \end{cases}$$

$$\because b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$\Rightarrow b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\sin\frac{n\pi x}{2}dx = \frac{k}{2}\left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_{-1}^{1} = 0$$

Thus Fourier series is $f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - +....\right)$

Example: A sinusoidal voltage $E_0 \sin\omega t$ where t is time is passed through half-wave rectifier that clips the negative portion of the wave. Find the Fourier series of the resulting periodic function



$$f(t) = \begin{cases} 0 & \text{if} \quad -\dfrac{\pi}{\omega} < t < 0 \\ E_0 \sin\omega t & \text{if} \quad 0 < t < \dfrac{\pi}{\omega} \end{cases} \qquad p = 2L = 2\frac{\pi}{\omega},\ L = \frac{\pi}{\omega}$$

Solution: Let $f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right)$

$$\because a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)dt$$

$$\Rightarrow a_0 = \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)dt = \frac{\omega}{2\pi}\int_{0}^{\pi/\omega} E_0\sin\omega t dt = \frac{\omega}{2\pi}\left[-\frac{E_0}{\omega}\cos\omega t\right]_0^{\pi/\omega} = \frac{E_0}{\pi}$$

$$\because a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

$$\Rightarrow a_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\cos n\omega t dt = \frac{\omega}{\pi}\int_{0}^{\pi/\omega} E_0\sin\omega t\cos n\omega t dt = \frac{E_0\omega}{2\pi}\int_{0}^{\pi/\omega} 2\sin\omega t\cos n\omega t dt$$

$$\Rightarrow a_n = \frac{E_0\omega}{2\pi}\int_{0}^{\pi/\omega}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt$$

For $n = 1$,

$$a_1 = \frac{E_0\omega}{2\pi}\int_{0}^{\pi/\omega}\sin 2\omega t dt = \frac{E_0\omega}{2\pi}\left[\frac{-\cos 2\omega t}{2\omega}\right]_0^{\pi/\omega} = 0$$

For $n = 2,3,4....,$

$$\Rightarrow a_n = \frac{E_0\omega}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$\Rightarrow a_n = \frac{E_0}{2\pi}\left[\frac{-\cos(1+n)\pi + 1}{(1+n)} + \frac{-\cos(1-n)\pi + 1}{(1-n)}\right] = \begin{cases} 0 & n = 3,5,7... \\ \dfrac{2E_0}{\pi(1-n^2)} & n = 2,4,6.... \end{cases}$$

$$\because b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$\Rightarrow b_n = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(t)\sin n\omega t dt = \frac{\omega}{\pi}\int_0^{\pi/\omega} E_0 \sin\omega t \sin n\omega t dt = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega} 2\sin\omega t\sin n\omega t dt$$

$$\Rightarrow b_n = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega}\left[-\cos(1+n)\omega t + \cos(1-n)\omega t\right]dt$$

For $n = 1$, $b_1 = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega}\left[1-\cos 2\omega t\right]dt = \frac{E_0\omega}{2\pi}\int_0^{\pi/\omega}\left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^{\pi/\omega} = \frac{E_0\omega}{2\pi}\frac{\pi}{\omega} = \frac{E_0}{2}$

For $n = 2,3,4....,$

$$\Rightarrow b_n = \frac{E_0\omega}{2\pi}\left[-\frac{\sin(1+n)\omega t}{(1+n)\omega} + \frac{\sin(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega} = \frac{E_0}{2\pi}\left[\frac{-\sin(1+n)\pi}{(1+n)} + \frac{\sin(1-n)\pi}{(1-n)}\right] = 0$$

Thus, Fourier series

$$f(t) = a_0 + b_1\sin\omega t + \sum_{n=2,4...}^{\infty} a_n\cos n\omega t = a_0 + b_1\sin\omega t + \sum_{n=2,4...}^{\infty}\frac{2E_0}{\pi(1-n^2)}\cos n\omega t$$

$$\Rightarrow f(t) = \frac{E_0}{\pi} + \frac{E_0}{2}\sin\omega t - \frac{2E_0}{\pi}\left[\frac{\cos 2\omega t}{3} + \frac{\cos 4\omega t}{15} + ...\right]$$

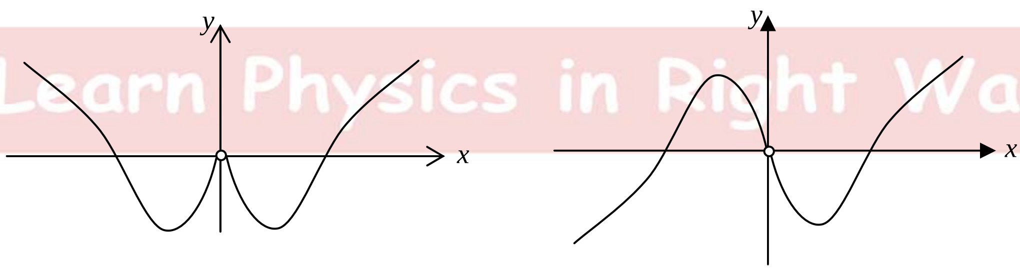
$$\Rightarrow f(t) = \frac{E_0}{\pi} + \frac{E_0}{2}\sin\omega t - \frac{2E_0}{\pi}\left[\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + ...\right]$$

5.4 Even and Odd functions and Half-Range Expansion

5.4.1 Even and Odd function

A function $g(x)$ is said to be even if $g(-x) = g(x)$, so that its graph is symmetrical with respect to vertical axis.

A function $h(x)$ is said to be even if $h(-x) = -h(x)$.



(a) Even function (b) Odd function

Since the definite integral of a function gives the area under the curve of the function between the limits of integration, we have

$$\int_{-L}^{L} g(x)dx = 2\int_0^L g(x)dx \qquad \text{for even } g(x)$$

$$\int_{-L}^{L} h(x)dx = 0 \qquad \text{for odd } h(x)$$

Fourier Cosine Series and Fourier Sine Series

Fourier series of an **even** function of period $2L$, is a **"Fourier cosine series"**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \qquad \ldots.(1)$$

with coefficients (note integration from 0 to L)

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad , \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad \ldots.(2)$$

Fourier series of an odd function of period $2L$, is a **"Fourier sine series"**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \qquad \ldots.(3)$$

with coefficients $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

NOTE:

(i) For even function $f(x)$;

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0$.

(ii) For odd function $f(x)$;

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 0, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = 0$$

and $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

The Case of Period 2π

If $L = \pi$ and $f(x)$ is even function, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots.(1')$$

with coefficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \quad , \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad \ldots.(2')$$

If $f(x)$ is odd function then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots.(3')$$

with coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \qquad \ldots.(4')$$

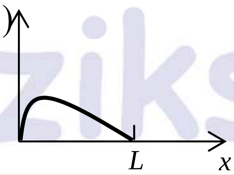
5.4.2 Half-Range Expansion

Half-range expansions are Fourier series. The idea is simple and useful. We could extend $f(x)$ as a function of period L and develop the extended function into a Fourier series. But this series would in general contain both cosine and sine terms.

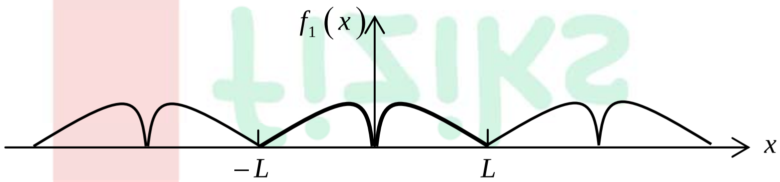
We can do better and get simpler series. For our given function $f(x)$, we can calculate **Fourier cosine series** coefficient (a_0 and a_n). This is the even periodic extension $f_1(x)$ of $f(x)$ in figure (b).

For our given function $f(x)$ we can calculate **Fourier sine series** coefficient (b_n). This is the odd periodic extension $f_2(x)$ of $f(x)$ in figure (c).

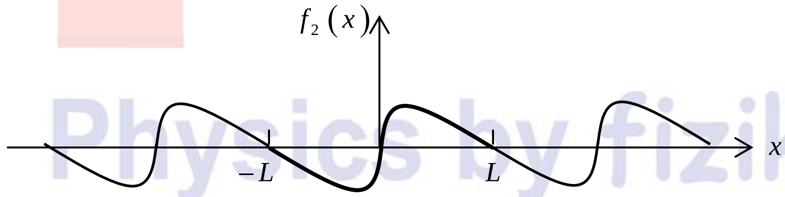
Both extensions have period $2L$. Note that $f(x)$ is given only on half the range, half the interval of periodicity of length $2L$.



(a) The given function



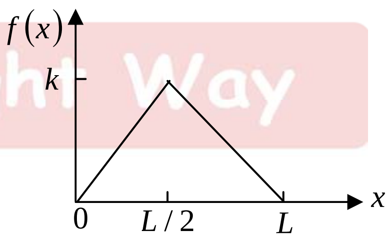
(b) $f(x)$ extended as an even periodic function of period $2L$



(c) $f(x)$ extended as an odd periodic function of period $2L$

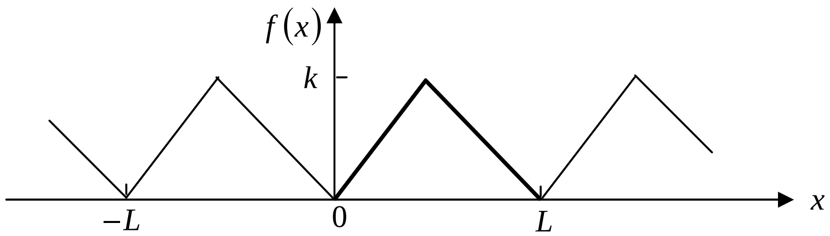
Example: Find the two half-range expansion of the function $f(x)$ as shown in figure below.

$$f(x)=\begin{cases}\dfrac{2k}{L}x & \text{if} \quad 0<x<\dfrac{L}{2}\\[2mm] \dfrac{2k}{L}(L-x) & \text{if} \quad \dfrac{L}{2}<x<L\end{cases}$$



Solution:

Even periodic extension



$\because a_0 = \frac{1}{L}\int_0^L f(x)\,dx$

$\Rightarrow a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\,dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\,dx\right] = \frac{k}{2}$

$\because a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \Rightarrow a_n = \frac{1}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\cos\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\cos\frac{n\pi x}{L}dx\right]$

Let us calculate the integral

$\int_0^{\frac{L}{2}} x\cos\frac{n\pi x}{L}dx = \frac{L}{n\pi}\left[x\sin\frac{n\pi x}{L}\right]_0^{L/2} + \left(\frac{L}{n\pi}\right)^2\left[\cos\frac{n\pi x}{L}\right]_0^{L/2}$

$\Rightarrow \int_0^{\frac{L}{2}} x\cos\frac{n\pi x}{L}dx = \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$

and $\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}dx = \frac{L}{n\pi}\left[(L-x)\sin\frac{n\pi x}{L}\right]_{L/2}^{L} - \left(\frac{L}{n\pi}\right)^2\left[\cos\frac{n\pi x}{L}\right]_{L/2}^{L}$

$\Rightarrow \int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}dx = -\frac{L^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$

$\Rightarrow a_n = \frac{1}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\cos\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\cos\frac{n\pi x}{L}dx\right] = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$

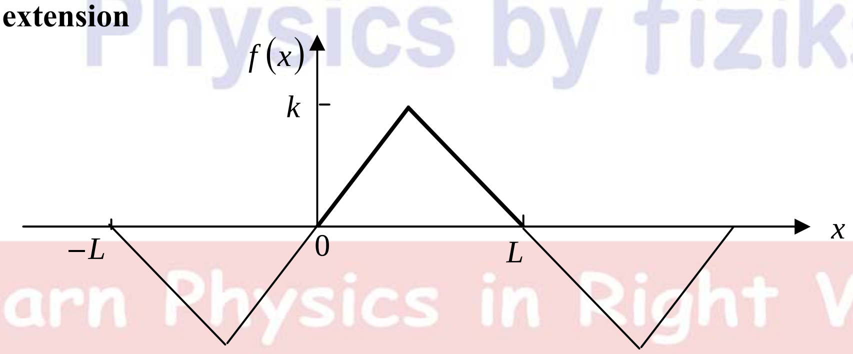
$\Rightarrow a_2 = -\frac{16k}{2^2\pi^2}, a_6 = -\frac{16k}{6^2\pi^2}, a_{10} = -\frac{16k}{10^2\pi^2}, \ldots\ldots\ldots$ and $a_n = 0$ if $n \neq 2, 6, 10\ldots..$

Hence the first half-range expansion of $f(x)$ is

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{L} + \frac{1}{6^2}\cos\frac{6\pi x}{L} + \frac{1}{10^2}\cos\frac{10\pi x}{L} + \ldots..\right)$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$ as shown in figure.

Odd periodic extension



$\because b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \Rightarrow b_n = \frac{2}{L}\int_0^{\frac{L}{2}} f(x)\sin\frac{n\pi x}{L}dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} f(x)\sin\frac{n\pi x}{L}dx$

$\Rightarrow b_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\sin\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\sin\frac{n\pi x}{L}dx\right]$

Let us calculate the integral

$\int_0^{\frac{L}{2}} x\sin\frac{n\pi x}{L}dx = \frac{L}{n\pi}\left[-x\cos\frac{n\pi x}{L}\right]_0^{L/2} + \left(\frac{L}{n\pi}\right)^2\left[\sin\frac{n\pi x}{L}\right]_0^{L/2}$

$$\Rightarrow \int_0^{\frac{L}{2}} x\cos\frac{n\pi x}{L}dx = -\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

and $\int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}dx = \frac{L}{n\pi}\left[-(L-x)\cos\frac{n\pi x}{L}\right]_{L/2}^{L} - \left(\frac{L}{n\pi}\right)^2\left[\sin\frac{n\pi x}{L}\right]_{L/2}^{L}$

$$\Rightarrow \int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}dx = \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\Rightarrow b_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{\frac{L}{2}} x\sin\frac{n\pi x}{L}dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L-x)\sin\frac{n\pi x}{L}dx\right] = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence the other half-range expansion of $f(x)$ is

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi x}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L} + \frac{1}{5^2}\sin\frac{5\pi x}{L} - + ...\right)$$

This Fourier sine series represents the odd periodic extension of the given function $f(x)$, of period $2L$ as shown in figure.

5.5 Complex Fourier Series

The Fourier series $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ ....(1)

can be written in complex form, which sometimes simplifies calculations.

$\because e^{inx} = \cos nx + i\sin nx$ and $e^{-inx} = \cos nx - i\sin nx$

$\Rightarrow \cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right)$ and $\sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$

Thus $a_n\cos nx + b_n\sin nx = \frac{1}{2}a_n\left(e^{inx} + e^{-inx}\right) + \frac{1}{2i}b_n\left(e^{inx} - e^{-inx}\right)$

$$\Rightarrow a_n\cos nx + b_n\sin nx = \frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx}$$

Lets take $a_0 = c_0$, $\frac{1}{2}(a_n - ib_n) = c_n$ and $\frac{1}{2}(a_n + ib_n) = k_n$, then (1) becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{inx} + k_n e^{-inx}\right) \quad ....(2)$$

where coefficients c_0, c_n and k_n are given by $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i\sin nx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$k_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i\sin nx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx$$

We can also write (2) as (take $k_n = c_{-n}$)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx; \quad n = 0, \pm 1, \pm 2.......$$

This is so called complex form of the Fourier series or, complex Fourier series of $f(x)$.

Here c_n are called complex Fourier series coefficients of $f(x)$.

For a function of period $2L$

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}} \quad \text{where } c_n=\frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi x}{L}}dx; \quad n=0,\pm1,\pm2.......$$

Example: Find the complex Fourier series of the periodic function

$f(x)=e^x; \quad (-\pi<x<\pi)$, having period 2π.

Solution: Let Fourier series is $f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx}$

$$\because c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \Rightarrow c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}dx=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}dx=\frac{1}{2\pi}\frac{1}{(1-in)}\left[e^{(1-in)x}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{2\pi}\frac{1}{(1-in)}\left[e^{(1-in)\pi}-e^{-(1-in)\pi}\right]$$

$$\Rightarrow c_n=\frac{1}{2\pi}\frac{1}{(1-in)}\left[e^{\pi}e^{-in\pi}-e^{-\pi}e^{in\pi}\right]=\frac{1}{2\pi}\left(\frac{1+in}{1+n^2}\right)\left[e^{\pi}-e^{-\pi}\right](-1)^n \qquad \because e^{\pm in\pi}=(-1)^n$$

$$\Rightarrow c_n=\left(\frac{1+in}{1+n^2}\right)\frac{\sinh\pi}{\pi}(-1)^n \qquad \because \sinh\pi=\frac{e^{\pi}-e^{-\pi}}{2}$$

Thus, Fourier series is $f(x)=\frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{1+in}{1+n^2}\right)e^{inx}$ ....(1)

Let us derive the real Fourier series

$\because (1+in)e^{inx}=(1+in)(\cos nx+i\sin nx)=(\cos nx-n\sin nx)+i(\cos nx+\sin nx)$

$\because n$ varies from $-\infty$ to $+\infty$, equation (1) has corresponding term with $-n$ instead of n.

Thus, $\because (1-in)e^{-inx}=(1-in)(\cos nx-i\sin nx)=(\cos nx-n\sin nx)-i(\cos nx+\sin nx)$

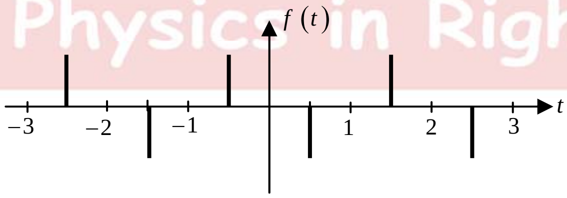
Let's add these two expressions; $(1+in)e^{inx}+(1-in)e^{-inx}=2(\cos nx-n\sin nx), \quad n=1,2,3......$

For $n=0$, $(-1)^n\left(\frac{1+in}{1+n^2}\right)e^{inx}=1$

Thus, $f(x)=\frac{\sinh\pi}{\pi}+2\frac{\sinh\pi}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{(\cos nx-n\sin nx)}{1+n^2}$

$$f(x)=\frac{2\sinh\pi}{\pi}\left[\frac{1}{2}-\frac{1}{1+1^2}(\cos x-\sin x)+\frac{1}{1+2^2}(\cos 2x-2\sin 2x)-+...\right]$$

Example: Consider the periodic function $f(t)$ with time period T as shown in the figure below.



The spikes, located at $t=\frac{1}{2}(2n-1)$, where $n=0,\pm1,\pm2,...,$ are Dirac-delta function of strength ±1. Find the amplitudes a_n in the Fourier expansion of $f(t)=\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n t/T}$.

Solution: $f(x)=\sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$, where $c_n=\frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi x}{L}}dx$; $n=0,\pm1,\pm2.......$

and Range: $[-1,1]$, hence $2L=2$.

Comparing with $f(t)=\sum_{n=-\infty}^{\infty} a_n e^{2\pi int/T}$, $L=\frac{T}{2}=1\Rightarrow T=2$.

$\Rightarrow f(t)=\sum_{n=-\infty}^{\infty} a_n e^{i\pi nt} \Rightarrow a_n=\frac{1}{2L}\int_{-L}^{L} f(t)e^{-i\pi nt}dt$

$\Rightarrow a_n=\frac{1}{2}\int_{-1}^{1} f(t)e^{-i\pi nt}dt=\frac{1}{2}\int_{-1}^{1}\left[\delta\left(t+\frac{1}{2}\right)-\delta\left(t-\frac{1}{2}\right)\right]e^{-i\pi nt}dt$

$\Rightarrow a_n=\frac{1}{2}\left[e^{-i\pi n(-1/2)}-e^{-i\pi n(1/2)}\right]=\frac{1}{2}\left[e^{i\pi n/2}-e^{-i\pi n/2}\right]=i\sin\frac{n\pi}{2}$

5.6 Approximation by Trigonometric Polynomials

Fourier series have major applications in approximation theory, that is, the approximation of functions by simpler functions.

Let $f(x)$ be a periodic function, of period 2π for simplicity that can be represented by a Fourier series. Then the N^{th} partial sum of the series is an approximation to $f(x)$

$$f(x)\approx a_0+\sum_{n=1}^{N}(a_n\cos nx+b_n\sin nx) \qquad ....(1)$$

We have to see whether (1) is the "best" approximation to f by a trigonometric polynomial of degree N, that is , by a function of the form

$$F(x)=A_0+\sum_{n=1}^{N}(A_n\cos nx+B_n\sin nx) \qquad ....(2)$$

where "best" means that the "error" of approximation is minimum.

The **total square error** of F relative to f on the interval $-\pi\le x\le\pi$ is given by

$$E=\int_{-\pi}^{\pi}(f-F)^2dx \qquad \text{clearly } E\ge 0.$$

The function F is a good approximation to f but $|f-F|$ is large at a point of discontinuity x_0.

5.6.1 Minimum square error

The total square error of F relative to f on the interval $-\pi\le x\le\pi$ is minimum if and only if the coefficients of $F(x)$ are the Fourier coefficients of $f(x)$. This minimum value E^* is given by

$$E^*=\int_{-\pi}^{\pi} f^2dx-\pi\left[2a_0^2+\sum_{n=1}^{N}(a_n^2+b_n^2)\right] \qquad ....(3)$$

From (3) we can see that E^* cannot increase as N increases, but may decrease. Hence with increasing N the partial sums of the Fourier series of f yields better and better approximations to f.

5.6.2 Parseval's Identity: Since $E^*\ge 0$ and equation (3) holds for every N, we obtain

$$2a_0^2+\sum_{n=1}^{N}(a_n^2+b_n^2)\le\frac{1}{\pi}\int_{-\pi}^{\pi} f^2dx$$

Now Parseval's Identity is $2a_0^2+\sum_{n=1}^{N}(a_n^2+b_n^2)=\frac{1}{\pi}\int_{-\pi}^{\pi} f^2dx$

H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

Example: Compute the total square error of F with $N = 3$ relative to

$$f(x) = x + \pi \qquad (-\pi < x < \pi)$$

on the interval $-\pi \le x \le \pi$.

Solution: Fourier coefficients are $a_0 = \pi$, $a_n = 0$ and $b_n = -\frac{2}{n}\cos n\pi$.

Its Fourier series is given by

$$F(x) = \pi + 2\sin x - \sin 2x + \frac{2}{3}\sin 3x.$$

Hence,

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)\right] = \int_{-\pi}^{\pi}(x+\pi)^2 dx - \pi\left[2\pi^2 + 2^2 + 1^2 + \left(\frac{2}{3}\right)^2\right]$$

$$E^* = \frac{8}{3}\pi^3 - \pi\left[2\pi^2 + \frac{49}{9}\right] \approx 3.567$$

Although $\left|f(x) - F(x)\right|$ is large at $x = \pm\pi$, where f is discontinuous, F approximates f quite well on the whole interval.