TIFR-2017 (Mathematical Physics Question and Solution)

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. Denote the commutator of two matrices A and B by $[A,B]=AB-BA$ and the anticommutator by $\{A,B\}=AB+BA$.

If $\{A,B\}=0$, we can write $[ABC]=$

(a) $-B[A,C]$ (b) $B\{A,C\}$ (c) $[A,C]B$ (d) $-B\{A,C\}$

Ans. : (d)

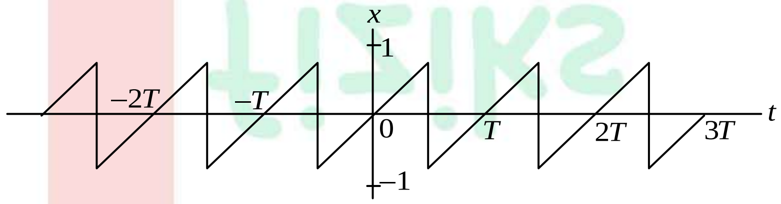
Solution: [A, BC] = ABC –

BCA $= ABC-BCA+BAC-BAC = ABC+BAC-(BCA+BAC)$

$$=(AB+BA)C-B(AC+CA)=\{A,B\}C-B\{A,C\}=-B\{A,C\}$$

Hence 'd' is the correct answer.

Q2. Consider the waveform $x(t)$ shown in the diagram below.



The Fourier series for $x(t)$ which gives closest approximation to this waveform is

(a) $x(t)=\frac{2}{\pi}\left[\cos\frac{\pi t}{T}-\frac{1}{2}\cos\frac{4\pi t}{T}+\frac{1}{3}\cos\frac{3\pi t}{T}+...\right]$

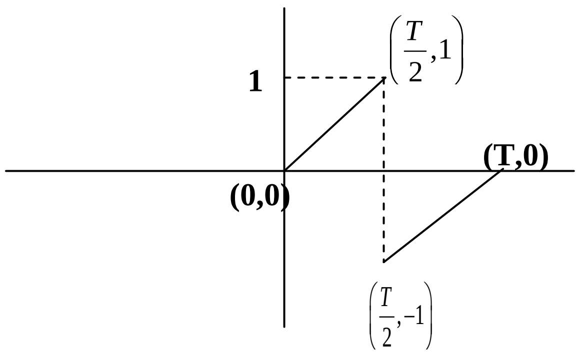
(b) $x(t)=\frac{2}{\pi}\left[-\cos\frac{2\pi t}{T}+\frac{1}{2}\cos\frac{4\pi t}{T}-\frac{1}{3}\cos\frac{6\pi t}{T}+...\right]$

(c) $x(t)=\frac{2}{\pi}\left[\sin\frac{\pi t}{T}-\frac{1}{2}\sin\frac{4\pi t}{T}+\frac{1}{3}\sin\frac{3\pi t}{T}+...\right]$

(d) $x(t)=\frac{2}{\pi}\left[-\sin\frac{\pi t}{T}+\frac{1}{2}\sin\frac{2\pi t}{T}-\frac{1}{3}\sin\frac{3\pi t}{T}+...\right]$

Ans. : (c)

Solution: The given wave form is from 0 to T



Let's find equation of the line with end points

$(0,0)$ and $\left(\frac{T}{2},1\right)$; $x-x_1=\frac{x_2-x_1}{t_2-t_1}(t-t_1)$

$$x-0=\frac{1-0}{\frac{T}{2}-0}(t-0);\quad x=\frac{2}{T}t,\ 0<t<\frac{T}{2}$$

$$x=\frac{2}{T}t,\ \frac{-T}{2}<t<0\ \text{as well}$$

$$a_0=\frac{2}{T}\int_{-T/2}^{T/2}x(t)\,dt=\frac{2}{T}\int_{-T/2}^{T/2}\frac{2t}{T}dt=\frac{4}{T^2}\left|\frac{t^2}{2}\right|_{-T/2}^{T/2}=\frac{4}{T}\left[\frac{T^2}{8}-\frac{T^2}{8}\right]=0$$

As $x(-t)=\frac{-2t}{T}=-x(t)$, function is odd, So $a_n=0$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2}\frac{2t}{T}\sin n\omega t\,dt=\frac{4}{T^2}\left[\left.\frac{-t\cos n\omega t}{n\omega}\right|_{-T/2}^{T/2}+\frac{1}{n^2\omega^2}\left|\sin n\omega t\right|_{-T/2}^{T/2}\right]$$

$$=\frac{4}{T^2}\left[-\frac{1}{n\omega}\left[\frac{T}{2}\cos\frac{n\omega T}{2}+\frac{T}{2}\cos\frac{n\omega T}{2}\right]+\frac{1}{n^2\omega^2}\left[\sin\frac{n\omega T}{2}+\sin\frac{n\omega T}{2}\right]\right]$$

$\sin\frac{n\omega T}{2}=\sin\frac{2n\pi}{2}=\sin(n\pi)=0$. Therefore second term vanishes

Thus, $b_n=\frac{-4T}{T^2 n\omega}\cos\frac{n\omega T}{2}=\frac{-4}{n\omega T}\cos n\pi=\frac{-4}{n2\pi}\cos n\pi=\frac{-2}{n\pi}\cos n\pi=\begin{bmatrix}\frac{-2}{n\pi},\ \textit{for even } n\\ \frac{2}{n\pi},\ \textit{for odd } n\end{bmatrix}$

Thus, $x(t)=a_0+\sum a_n\cos n\omega t+\sum b_n\sin n\omega t$

$$x(t)=\sum b_n\sin\frac{2n\pi t}{T}=b_1\sin\frac{2\pi t}{T}+b_2\sin\frac{4\pi t}{T}+b_3\sin\frac{6\pi t}{T}$$

$$b_1=\frac{2}{\pi},\ b_2=\frac{-2}{2\pi},\ b_3=\frac{2}{3\pi},\ b_4=\frac{-2}{4\pi}$$

$$x(t)=\frac{2}{\pi}\sin\frac{2\pi t}{T}-\frac{2}{2\pi}\sin\frac{4\pi t}{T}+\frac{2}{3\pi}\sin\frac{6\pi t}{T}+\ldots..$$

$$x(t)=\frac{2}{\pi}\left[\sin\frac{2\pi t}{T}-\frac{1}{2}\sin\frac{4\pi t}{T}+\frac{1}{3}\sin\frac{6\pi t}{T}-\frac{1}{4}\sin\frac{8\pi t}{T}+\ldots...\right]$$

Q3. The matrix

$$\begin{pmatrix}100\sqrt{2} & x & 0\\ -x & 0 & -x\\ 0 & x & 100\sqrt{2}\end{pmatrix}$$

where $x>0$, is known to have two equal eigenvalues. Find the value of x.

Ans. : 50

H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

Solution:

Given that the matrix has two eigen values

Let's find the eigen values

$$\begin{pmatrix} 100\sqrt{2} & x & 0 \\ -x & 0 & -x \\ 0 & x & 100\sqrt{2} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 100\sqrt{2}-\lambda & x & 0 \\ -x & -\lambda & -x \\ 0 & x & 100\sqrt{2}-\lambda \end{pmatrix} = 0$$

$$\Rightarrow 100\sqrt{2}-\lambda \begin{vmatrix} -\lambda & -x \\ x & 100\sqrt{2}-\lambda \end{vmatrix} - x \begin{vmatrix} -x & -x \\ 0 & 100\sqrt{2}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(100\sqrt{2}-\lambda\right)\left[\lambda^2 - 100\sqrt{2}\lambda + x^2\right] + x^2\left[100\sqrt{2}-\lambda\right] = 0$$

$$\Rightarrow \left(100\sqrt{2}-\lambda\right)\left[\lambda^2 - 100\sqrt{2}\lambda + x^2\right] = 0$$

One of the eigen values = $100\sqrt{2}$

The other two eigen values are given by solution of

$$\lambda^2 - 100\sqrt{2}\lambda + 2x^2 = 0 \qquad ...(1)$$

Roots of (1) will be equal if $b^2 - 4ac = 0$

$$\Rightarrow \left(100\sqrt{2}\right)^2 - 4\times 2\times x^2 = 0 \Rightarrow \cancel{8}x^2 = \cancel{100}\times 100 \times \cancel{2}$$

$$\Rightarrow x^2 = 2500 \Rightarrow x = 50$$

The two eigen values will be $\lambda = \dfrac{100\sqrt{2} \pm \sqrt{0}}{2} = 50\sqrt{2},\ 50\sqrt{2}$

value of x = 50

SECTION B- (only for Int.-Ph.D. candidates)

Q4. A unitary matrix U is expanded in terms of a Hermitian matrix H, such that $U = e^{\frac{i\pi H}{2}}$

if we know that $H = \begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix}$, then U must be

(a) $\begin{pmatrix} \frac{i}{2} & 0 & \frac{i\sqrt{3}}{2} \\ 0 & i & 0 \\ \frac{i\sqrt{3}}{2} & 0 & -\frac{i}{2} \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 2 & 0 \\ \sqrt{3} & 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} i & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & i & \frac{1}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & i \end{pmatrix}$ (d) $\begin{pmatrix} 2i & 1 & \frac{\sqrt{3}}{2} \\ 1 & 2i & 0 \\ \frac{\sqrt{3}}{2} & 0 & 2i \end{pmatrix}$

Ans. : (a)

Solution:

If a matrix is unitary, $uu^{\dagger} = I$

Let's test this relation for every possible solution.

(a) $$\begin{bmatrix} \frac{i}{2} & 0 & i\frac{\sqrt{3}}{2} \\ 0 & i & 0 \\ \frac{i\sqrt{3}}{2} & 0 & \frac{-i}{2} \end{bmatrix}\begin{bmatrix} \frac{-i}{2} & 0 & \frac{-i\sqrt{3}}{2} \\ 0 & -i & 0 \\ \frac{-i\sqrt{3}}{2} & 0 & \frac{i}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}+\frac{3}{4} & 0 & \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} & 0 & \frac{3}{4}+\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}$$

(b)

$$\begin{bmatrix} 1 & 0 & \sqrt{3} \\ 0 & 2 & 0 \\ \sqrt{3} & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & \sqrt{3} \\ 0 & 2 & 0 \\ \sqrt{3} & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1+0+3 & 0+0+0 & \sqrt{3}+0-\sqrt{3} \\ 0+0+0 & 0+4+0 & 0+0+0 \\ \sqrt{3}+0-\sqrt{3} & 0+0+0 & 3+0+1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \neq I$$

(c) $$\begin{pmatrix} i & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & i & \frac{1}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & i \end{pmatrix}\begin{pmatrix} -i & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & -i & \frac{1}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -i \end{pmatrix} = \begin{bmatrix} 1+\frac{1}{4}+\frac{3}{4} & \frac{i}{2}-\frac{i}{2}+\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{2}i+\frac{1}{4}-\frac{\sqrt{3}}{2}i \\ \frac{-i}{2}+\frac{i}{2}+\frac{\sqrt{3}}{4} & \frac{1}{4}+1+\frac{1}{4} & \frac{\sqrt{3}}{4}+\frac{1}{2}-\frac{1}{2} \\ \frac{-\sqrt{3}}{2}i+\frac{1}{4}+\frac{\sqrt{3}}{2}i & \frac{\sqrt{3}}{4}-\frac{i}{2}+\frac{i}{2} & \frac{3}{4}+\frac{1}{4}+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & \frac{\sqrt{3}}{4} & \frac{1}{4} \\ \frac{\sqrt{3}}{4} & 2 & \frac{\sqrt{3}}{4} \\ \frac{1}{4} & \frac{\sqrt{3}}{4} & 2 \end{bmatrix} \neq I$$

(d) $\begin{pmatrix} 2i & 1 & \frac{\sqrt{3}}{2} \\ 1 & 2i & 0 \\ \frac{\sqrt{3}}{2} & 0 & 2i \end{pmatrix}\begin{pmatrix} -2i & 1 & \frac{\sqrt{3}}{2} \\ 1 & -2i & 0 \\ \frac{\sqrt{3}}{2} & 0 & -2i \end{pmatrix}$

$$=\begin{bmatrix} 4+1+\frac{3}{4} & 2i-2i+0 & \sqrt{3}i+0-\sqrt{3}i \\ -2i+2i+0 & 1+4+0 & \frac{\sqrt{3}}{2}+0-0 \\ -\sqrt{3}i+\sqrt{3}i & \frac{\sqrt{3}}{2}+0+0 & \frac{3}{4}+0+4 \end{bmatrix}=\begin{bmatrix} \frac{23}{4} & 0 & 0 \\ 0 & 5 & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{19}{4} \end{bmatrix}\neq I$$

Hence (a) is the correct answer.

Q5. Evaluate the expression

$$n!\int_0^{x_n} dx_{n-1}\int_0^{x_{n-1}} dx_{n-2}\int_0^{x_{n-2}} dx_{n-3\ldots}\int_0^{x_3} dx_2\int_0^{x_2} dx_1\int_0^{x_1} dx_0$$

Ans. : (a)

Solution:

$$\int_0^{x_1} dx_0 = |x_0|_0^{x_1} = x_1 = \frac{x_1}{1!}$$

$$\int_0^{x_2} dx_1\int_0^{x_1} dx_0 = \int_0^{x_2} x_1 dx_1 = \left|\frac{x_1^2}{2}\right|_0^{x_2} = \frac{x_2^2}{1.2} = \frac{x_2^3}{2!}$$

$$\int_0^{x_3} dx_2\int_0^{x_2} dx_1\int_0^{x_1} dx_0 = \int_0^{x_3}\frac{x_2^2 dx_2}{1\cdot 2} = \frac{1}{1\cdot 2\cdot 3}\int_0 x_2^3\Big|_0^{x_3} = \frac{x_3^3}{3!}$$

$$\int_0^{x_4} dx_3\int_0^{x_3} dx_2\int_0^{x_2} dx_1\int_0^{x_1} dx_0 = \int_0^{x_4}\frac{x_3^3}{3!}dx_3 = \frac{1}{4!}|x_3^4|_0^{x_4} = \frac{x_4^4}{4!}$$

$$\int_0^{x_5}\frac{x_4^4}{4!}dx_4 = \frac{x_9^5}{5!}$$

Similarly, if we repeat the procedure, we will get $I = n!\times\frac{A^n}{n!} = A^n$

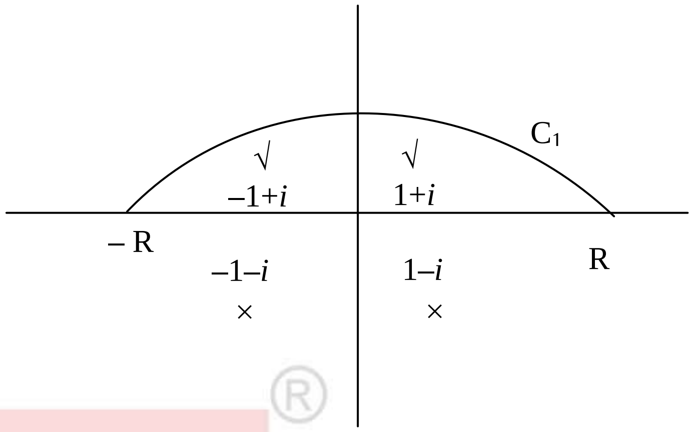
SECTION B-(Only for Ph.D. candidates)

Q6. The value of the integral $\int_0^\infty \frac{dx}{x^4+4}$ is

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

Ans.: **(a)**

Solution:



Consider $\int_{-\infty}^{\infty}\frac{dx}{x^4+4}$ and

$$f(z)=\frac{1}{z^4+4}$$

Poles are given by $z^4+4=0$

$$z^4=-4=4e^{\pi}\left[e^{\pi}=-1\right];$$

$$z=4^{1/4}e^{\frac{(2n+1)\pi}{4}}$$

$$z=\sqrt{2}\,e^{i\left(\frac{2n+1}{4}\right)\pi}\qquad\{n=0,1,2,3\}$$

$$z_1=\sqrt{2}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]=\sqrt{2}\left[\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right]=1+i=\sqrt{2}e^{\frac{i\pi}{4}}$$

$$z_2=\sqrt{2}\left[\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right]=\sqrt{2}\left[-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right]=-1+i=\sqrt{2}e^{\frac{2\pi}{4}}$$

$$z_3=-1-i,\quad z_4=1-i$$

z_3 and z_4 are located in the lower half complex plane. Hence, they do not contribute.

Let's calculate residue at $z=z_1$ and $z=z_2$.

z_1 : Residue

$$\frac{1}{\frac{d}{dz}\left(z^4+4\right)\Big|_{z=z_1}}=\frac{1}{4z^3\big|_{z=z_1}}=\frac{1}{4\left(\sqrt{2}\right)^3e^{\frac{3i\pi}{4}}}=\frac{1}{8\sqrt{2}}e^{-\frac{3i\pi}{4}}=\frac{1}{8\sqrt{2}}\left[\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right]$$

$$=\frac{1}{8\sqrt{2}}\left[\frac{-1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right]=\frac{-1}{16}(1+i)$$

z_2 : Residue $\frac{1}{4z^3\big|_{z=z_2}}=\frac{1}{4\left(\sqrt{2}e^{\frac{i3\pi}{4}}\right)^3}=\frac{1}{8\sqrt{2}}e^{\frac{-i9\pi}{4}}=\frac{1}{8\sqrt{2}}\left[\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right]=\frac{1}{16}[1-i]$

Thus, $\int_C\frac{dz}{z^4+4}=2\pi i\ \Sigma\ \text{Residue}=\int_{-R}^{R}\frac{dx}{x^4+4}+\int_{C_1}f(z)\,dz=2\pi i\,\Sigma\ \text{Residue}$

Take limit $R \to \infty$ and note that, limit $R \to \infty \int_{C_1} f(z)\,dz = 0$

$$+\int_{-\infty}^{\infty} \frac{dx}{x^4+4} + 0 = 2\pi i\left[\frac{1}{16}(-1-i) + \frac{1}{16}(1-i)\right]$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+4} = \frac{2\pi i}{16}\left[-1-i+1-i\right] = \frac{-4\pi i^2}{16} = \frac{\pi}{4}$$

Now $\int_{0}^{\infty} \frac{dx}{x^4+4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+4} = \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}$

Hence the answer is (a)

Q7. Write down $x(t)$, where $x(t)$ is the solution of the following differential equation

$$\left(\frac{d}{dt}+2\right)\left(\frac{d}{dt}+1\right)x = 1,$$

with the boundary conditions

$$\left.\frac{dx}{dt}\right|_{t=0} = 0, \qquad \left.x(t)\right|_{t=0} = -\frac{1}{2}$$

Ans.:

Solution:

$$\left(\frac{d}{dt}+2\right)\left(\frac{dx}{dt}+x\right) = 1 \Rightarrow \frac{d}{dt}\left[\frac{dx}{dt}+x\right] + 2\left[\frac{dx}{dt}+x\right] = 1 \Rightarrow \frac{d^2x}{dt^2} + 1 + 2\frac{dx}{dt} + 2x = 1$$

$$\Rightarrow \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2 = 0$$

Auxiliary equation is given by $D^2 + 2D + 2 = 0$

$$D = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

Thus, $x(t) = e^{-t}\left[A\cos t + B\sin t\right]$

$$x(0) = -\frac{1}{2} = e^{-0}\left[A\cos 0 + B\cdot 0\right] = -\frac{1}{2} = 1\left[A+0\right] \Rightarrow A = -\frac{1}{2}$$

$$x'(t) = -e^{-t}\left[A\cos t + B\sin t\right] + e^{-t}\left[-A\sin t + B\cos t\right]$$

$$\left.x'(t)\right|_{t=0} = 0 = -e^{-0}\left[A\cos 0 + B\sin 0\right] + e^{-0}\left[-A\sin 0 + B\cos 0\right]$$

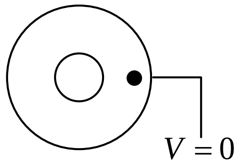
$$0 = e^{-0}\left[-A+B\right] \Rightarrow B = A \Rightarrow A = B = -\frac{1}{2}$$

Thus $x(t) = -\frac{1}{2}e^{-t}\left[\cos t + \sin t\right]$

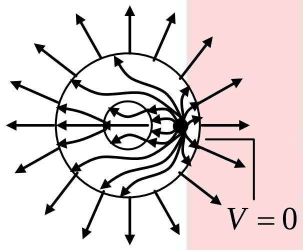
TIFR-2017 (EMT Question and Solution)

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

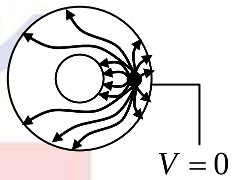
Q1. Two long hollow conducting cylinders, each of height h, are placed concentrically on the ground, as shown in the figure (top view). The outer cylinder is grounded, while the inner cylinder is insulated. A positive charge (the black dot in the figure) is placed between the cylinders at a height $\frac{h}{2}$ from the ground. Which of the following figures gives the most accurate representation (top view) of the lines of force?



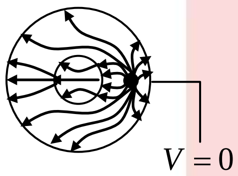
(a)



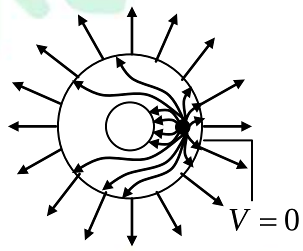
(b)



(c)



(d)



Ans. : (b)

Solution. :

Electric field can not penetrate the conductor.

Q2. A common model for the distribution of charge in a hydrogen atom has a point-like proton of charge $+q_0$ at the centre and an electron with a static charge density distribution $\rho(r) = -\frac{q_0}{\pi a^3} e^{\frac{-2r}{a}}$ where a is a constant. The electric field $\vec{E}$ at $r = a$ due to this system of charges will be

(a) $-\frac{5q_0}{4\pi \in_0 e^2 a^2}\hat{r}$

(b) $-\frac{5q_0}{4\pi \in_0 e a^2}\hat{r}$

(c) $\frac{3q_0}{4\pi \in_0 e^2 a^2}\hat{r}$

(d) $\frac{5q_0}{4\pi \in_0 e^2 a^2}\hat{r}$

Ans. : (d)

H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

Solution. :

$$\oint_S \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0}Q_{enc} \quad \Rightarrow \left|\vec{E}\right| \times 4\pi a^2 = \frac{1}{\varepsilon_0}\left[q_0 + \int_0^a \left(-\frac{q_0}{\pi a^3}e^{\frac{-2r}{a}}\right)4\pi r^2 dr\right]$$

$$\Rightarrow \left|\vec{E}\right| \times 4\pi a^2 = \frac{1}{\varepsilon_0}\left[q_0 - \frac{4q_0}{a^3}\int_0^a r^2 e^{\frac{-2r}{a}}dr\right] = \frac{1}{\varepsilon_0}\left[q_0 - \frac{4q_0}{a^3}\int_0^a r^2 e^{-\alpha r}dr\right] \text{ where } \alpha = \frac{2}{a}.$$

$$\because \int_0^a r^n e^{-\alpha r}dr = \frac{\lfloor n}{\alpha^{n+1}}\left[1 - e^{-\alpha a}\sum_{i=0}^{n}\frac{(\alpha a)^i}{\lfloor i}\right]$$

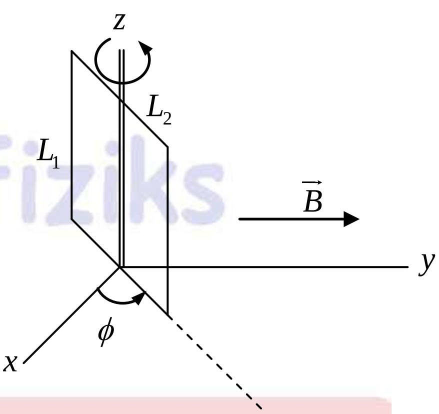
Thus $\int_0^a r^2 e^{-\alpha r}dr = \frac{\lfloor 2}{\alpha^3}\left[1 - e^{-\alpha a}\sum_{i=0}^{2}\frac{(\alpha a)^i}{\lfloor i}\right] = \frac{2}{\alpha^3}\left[1 - e^{-\alpha a}\left\{1 + \alpha a + \frac{(\alpha a)^2}{\lfloor 2}\right\}\right]$

$$\Rightarrow \int_0^a r^2 e^{-\alpha r}dr = 2 \times \frac{a^3}{8}\left[1 - e^{-\frac{2}{a}a}\left\{1 + \frac{2}{a}a + \frac{4}{a^2}\frac{a^2}{2}\right\}\right] = \frac{a^3}{4}\left[1 - 5e^{-2}\right]$$

$$\Rightarrow \left|\vec{E}\right| \times 4\pi a^2 = \frac{1}{\varepsilon_0}\left[q_0 - \frac{4q_0}{a^3}\left\{\frac{a^3}{4}\left(1 - 5e^{-2}\right)\right\}\right] = \frac{1}{\varepsilon_0}\left[q_0 - q_0\left(1 - 5e^{-2}\right)\right] = \frac{1}{\varepsilon_0}\left[5q_0 e^{-2}\right]$$

$$\Rightarrow \vec{E} = \frac{5q_0}{4\pi\varepsilon_0 e^2 a^2}\hat{r}$$

Q3. A rectangular metallic loop with sides L_1 and L_2 is placed in the vertical plane, making an angle ϕ with respect to the x-axis, as shown in the figure, and a spatially uniform magnetic field $\vec{B} = B\hat{y}$ is applied. The loop is free to rotate about the $\hat{z}$ axis (shown in the figure with a double line). The magnetic field changes with time at a constant rate



$$\frac{dB}{dt} = \kappa$$

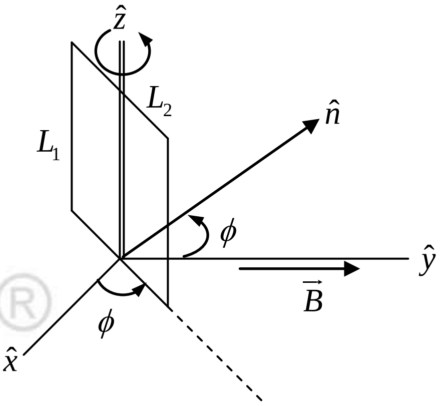
If the resistance of the loop is R, the torque τ required to prevent the loop from rotating will be

(a) $\kappa B\frac{(L_1L_2)^2}{2R}\sin\phi\hat{z}$ (b) $-\kappa B\frac{(L_1L_2)^2}{2R}\sin 2\phi\hat{z}$

(c) $\kappa B\frac{(L_1L_2)^2}{R}\sin\phi\cos\phi\hat{z}$ (d) $-\kappa B\frac{(L_1L_2)^2}{R}\sin\phi\hat{z}$

Ans. : (b)

Solution. : The torque τ required to prevent the loop from rotating will be $\vec{\tau} = \vec{m} \times \vec{B}$.



Magnetic flux through the loop is

$$\phi_m = \int_S \vec{B}.d\vec{a} = (B\hat{y}).(L_1L_2\hat{n}) = L_1L_2B\cos\phi$$

Induced emf in the loop

$$|\varepsilon| = \frac{d\phi_m}{dt} = L_1L_2\cos\phi\frac{dB}{dt} = L_1L_2\cos\phi\kappa$$

Induced current in the loop $I_{ind} = \frac{|\varepsilon|}{R} = \frac{L_1L_2\cos\phi\kappa}{R}$.

Magnetic dipole moment of the current loop is $\vec{m} = I_{ind} \times (L_1L_2)\hat{n} = -\frac{(L_1L_2)^2\cos\phi\kappa}{R}\hat{n}$.

$$\vec{\tau} = \frac{(L_1L_2)^2\cos\phi\kappa}{R}\hat{n} \times (B\hat{y}) = -\frac{(L_1L_2)^2\cos\phi\kappa}{R}(B\sin\phi\hat{z}) = -\kappa B\frac{(L_1L_2)^2}{2R}\sin 2\phi\hat{z}$$

SECTION B- (only for Int.-Ph.D. candidates)

Q4. An electromagnetic wave in free space is described by

$$\vec{E}(x,y,z,t) = \hat{z}\,E_0\cos\frac{1}{2}\left(kx - \sqrt{3}ky - 2\omega t\right)$$

The Poynting vector associated with this wave is along the direction

(a) $\hat{x} + \sqrt{3}\hat{y}$ (b) $\sqrt{3}\hat{x} + \hat{y}$ (c) $\hat{x} - \sqrt{3}\hat{y}$ (d) $-\sqrt{3}\hat{x} + \hat{y}$

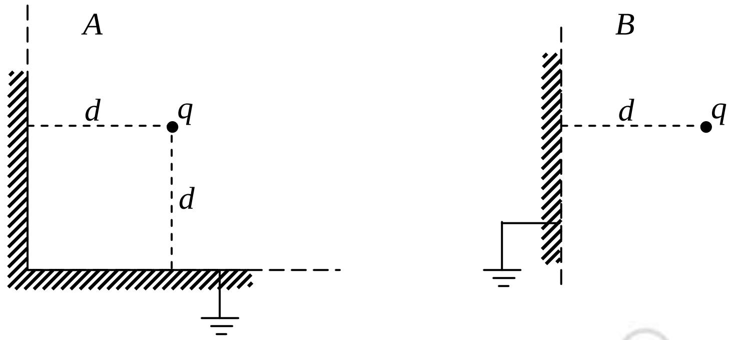
Ans. : (c)

Solution. : $\vec{E}(x,y,z,t) = \hat{z}\,E_0\cos\left(\vec{k}.\vec{r} - \omega t\right)$

$$\vec{E}(x,y,z,t) = \hat{z}\,E_0\cos\frac{1}{2}\left(kx - \sqrt{3}ky - 2\omega t\right) \Rightarrow \vec{k} = \frac{k}{2}\left(\hat{x} - \sqrt{3}\hat{y}\right)$$

Poynting vector is in the direction of $\vec{k}$.

Q5. Consider the following situations.

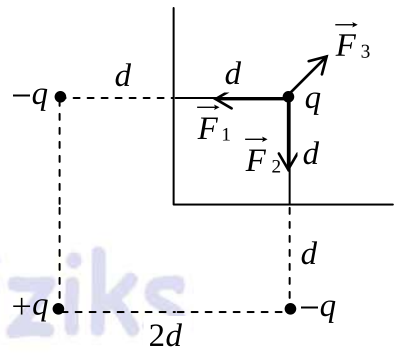


In situation A, two semi-infinite earthed conducting planes meet at right-angles. A point charge q, is placed at a distance d from each plane, as shown in the figure A. The magnitude of the force exerted on the charge q is denoted F_A.

In situation B, the same charge q is kept at the same distance d from an infinite earthed conducting plane, as shown in the figure B. The magnitude of the force exerted on the charge q is denoted F_B. Find the numerical ratio $\frac{F_A}{F_B}$

Ans. :

Solution. : $\left|\vec{F}_1\right| = \left|\vec{F}_2\right| = k\frac{q^2}{4d^2}$ and $\left|\vec{F}_3\right| = k\frac{q^2}{\left(2\sqrt{2}d\right)^2} = k\frac{q^2}{8d^2}$

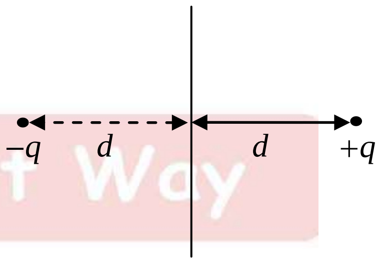


Resultant of $\vec{F}_1, \vec{F}_2$ is $F_{12} = \sqrt{F_1^2 + F_2^2} = \sqrt{2}k\frac{q^2}{4d^2}$.

Net force $\vec{F}_A = k\frac{q^2}{4d^2}\left(\sqrt{2} - \frac{1}{2}\right) = \frac{q^2}{32\pi\varepsilon_0 d^2}\left(2\sqrt{2} - 1\right)$

(towards the corner)

$$F_B \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d^2} = \frac{q^2}{16\pi\varepsilon_0 d^2}$$

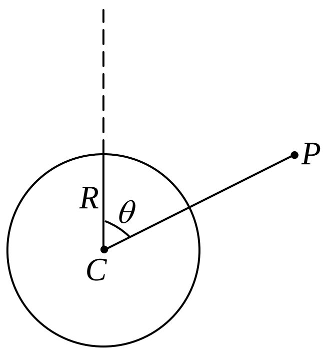


$$\frac{F_A}{F_B} = \frac{q^2}{32\pi\varepsilon_0 d^2}\left(2\sqrt{2} - 1\right) \times \frac{16\pi\varepsilon_0 d^2}{q^2} = \frac{\left(2\sqrt{2} - 1\right)}{2} = 0.9142$$

Q6. Consider a spherical shell with radius R such that the potential on the surface of the shell in spherical coordinates is given by,

$$V\left(r = R, \theta, \phi\right) = V_0 \cos^2\theta$$

where the angle θ is shown in the figure. There are no charges except for those on the shell. The potential outside the shell at the point P a distance $2R$ away from its center C (see figure) is



(a) $V=\frac{V_0}{8}\left(1+2\cos^2\theta\right)$ (b) $V=\frac{V_0}{4}\left(1-\cos^2\theta\right)$

(c) $V=\frac{V_0}{8}\left(1+\cos^2\theta\right)$ (d) $V=\frac{V_0}{2}\left(-2\cos\theta+\cos^3\theta\right)$

Ans. : (c)

Solution. :

$$V(R,\theta)=\sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta)=V_0(\theta)$$

$$\Rightarrow \frac{B_0}{R}P_0(\cos\theta)+\frac{B_1}{R^2}P_1(\cos\theta)+\frac{B_2}{R^3}P_2(\cos\theta)=V_0(\theta)$$

$$\frac{B_0}{R}+\frac{B_1}{R^2}\cos\theta+\frac{B_2}{R^3}\left(\frac{3\cos^2\theta-1}{2}\right)=V_0\cos^2\theta$$

$$\Rightarrow\left(\frac{B_0}{R}-\frac{B_2}{2R^3}\right)+\frac{B_1}{R^2}\cos\theta+\frac{3}{2}\frac{B_2}{R^3}\cos^2\theta=V_0\cos^2\theta$$

$$B_1=0,\ \frac{3}{2}\frac{B_2}{R^3}=V_0\Rightarrow B_2=\frac{2}{3}V_0R^3,\ \left(\frac{B_0}{R}-\frac{B_2}{2R^3}\right)=0\Rightarrow B_0=\frac{B_2}{2R^2}=\frac{1}{3}V_0R$$

Thus $B_0=\frac{1}{3}V_0R,\ B_1=0,\ B_2=\frac{2}{3}V_0R^3$

Potential outside $r>R$ is

$$V(r,\theta)=\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)=\frac{B_0}{r}P_0(\cos\theta)+\frac{B_1}{r^2}P_1(\cos\theta)+\frac{B_2}{r^3}P_2(\cos\theta)$$

$$V(2R,\theta)=\frac{1}{2R}\times\frac{1}{3}V_0R\times1+0+\frac{1}{(2R)^3}\times\frac{2}{3}V_0R^3\times\left(\frac{3\cos^2\theta-1}{2}\right)$$

$$\Rightarrow V(2R,\theta)=\frac{1}{6}V_0+\frac{2}{24}V_0\times\frac{3\cos^2\theta}{2}-\frac{2}{24}V_0\times\frac{1}{2}\Rightarrow V(2R,\theta)=\frac{1}{6}V_0+\frac{1}{8}V_0\cos^2\theta-\frac{1}{24}V_0$$

$$\Rightarrow V(2R,\theta)=\frac{V_0}{8}\left(1+\cos^2\theta\right)$$

Revised Edition-2022
H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

TIFR-2017 (Quantum Mechanics Question and Solution)

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. Denote the commutator of two matrices A and B by $[A,B]=AB-BA$ and the anticommutator by $\{A,B\}=AB+BA$.

If $\{A,B\}=0$, we can write $[ABC]=$

(a) $-B[A,C]$ (b) $B\{A,C\}$ (c) $[A,C]B$ (d) $-B\{A,C\}$

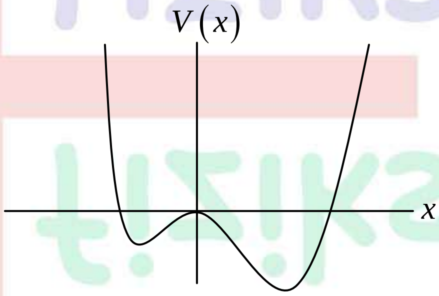
Ans. : (d)

Solution.: $[A,BC]=B[A,C]+[A,B]C=B[A,C]+(AB-BA)C$

$$=B(AC-CA)+(AB+BA-2BA)C=B(AC-CA)+(\{A,B\}-2BAC)$$

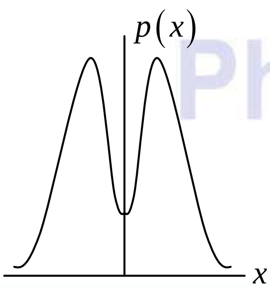
$$=B(AC-CA-2AC)=-B(AC+CA)=-B\{A,C\}$$

Q2. Consider the $1-D$ asymmetric double well potential $V(x)$ as sketched below.

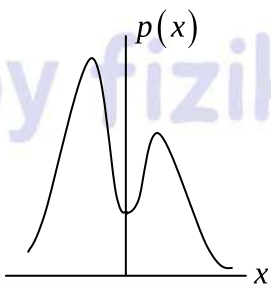


The probability distribution $p(x)$ of a particle in the ground state of this potential is best represented by

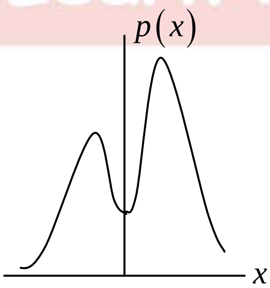
(a)



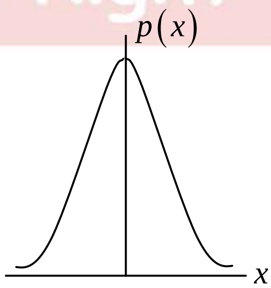
(b)



(c)



(d)



Ans. : (c)

Solution. :

This problem is solved through elimination method.

(a) Since the potential of the particle is not symmetrical, so the probability density of the particle is also not symmetrical. This eliminates option (a) & (d).

(b) The depth of well in the region $x<0$ is lower than the depth of well in the region $x>0$, it implies that the probability of finding the particle in the region $x<0$ is lower than finding the particle in region $x>0$.

(c) This is correctly depicted in the option (c).

Q3. The normalized wave function of a particle can be written as

$$\psi(x)=N\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{7}}\right)^n \varphi_n(x)$$

where $\varphi_n(x)$ are the normalized energy eigenfunctions of a given Hamiltonian. The value of N is

(a) $\sqrt{\frac{(6-2\sqrt{7})}{7}}$ (b) $\sqrt{\frac{1}{7}}$ (c) $\sqrt{\frac{3}{7}}$ (d) $\sqrt{\frac{6}{7}}$

Ans. : (d)

Solution. :

The normalized wave function of a particle can be written as

$$\psi(x)=N\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{7}}\right)^n \phi_n(x)=N\left(\phi_0(x)+\frac{1}{\sqrt{7}}\phi_1(x)+\frac{1}{\sqrt{7}}\phi_2(x)+\dots\right)$$

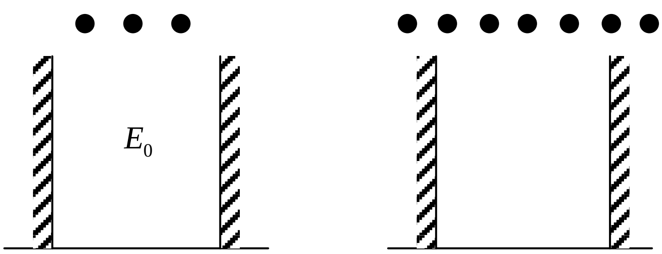
Applying the condition of normalization, we have

$$\langle\psi(x)|\psi(x)\rangle=N^2\left(\langle\phi_0(x)|\phi_0(x)\rangle+\frac{1}{7}\langle\phi_1(x)|\phi_1(x)\rangle+\frac{1}{7^2}\langle\phi_2(x)|\phi_2(x)\rangle+\dots\dots\right)=1$$

$$N^2\left(1+\frac{1}{7}+\frac{1}{7^2}+..\right)=1\Rightarrow N^2\left(\frac{1}{1-1/7}\right)=1\Rightarrow N^2\left(\frac{7}{6}\right)=1\ \Rightarrow N=\sqrt{\frac{6}{7}}$$

The value of N is $\sqrt{6/7}$.

Q4. A quantum mechanical system consists of a one-dimensional infinite box, as indicated in the figures below.



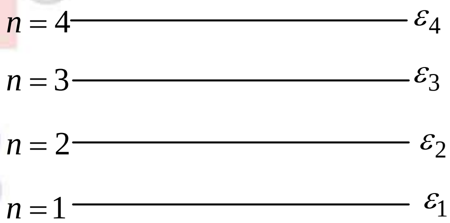
3 (three) identical non-interacting spin-$1/2$ particles; are first placed in the box, and the ground state energy of the system is found to be $E_0 = 18eV$. If 7 (seven) such identical particles are placed in the box, what will be the ground state energy, in units of eV?

Ans. :

Solution. :

$$E_2^{'} = 2E_1 + 2E_2 = \frac{2\pi^2\hbar^2}{2ma^2} + \frac{2^2\pi^2\hbar^2}{2ma^2} = \frac{6\pi^2\hbar^2}{2ma^2} = 18 \quad \Rightarrow \frac{\pi^2\hbar^2}{2ma^2} = 3eV$$

Now the seven particles are placed in the box, then the distribution of fermion in ground state is shown below i.e.,



The ground state energy in this state is by

$$E_1^1 = 2E_1 + 2E_2 + 2E_3 + E_4$$

$$= \frac{2\pi^2\hbar^2}{2ma^2} + \frac{2.4\pi^2\hbar^2}{2ma^2} + \frac{29\pi^2\hbar^2}{2ma^2} + \frac{16\pi^2\hbar^2}{2ma^2} = \frac{44\pi^2\hbar^2}{2ma^2}$$

$$E_1^1 = 44\frac{\pi^2\hbar^2}{2ma^2} = 44\times 3eV = 132eV$$

SECTION B- (only for Int.-Ph.D. candidates)

Q5. Electrons in a given system of hydrogen atoms are described by the wave function

$$\psi(r,\theta,\varphi) = 0.8\psi_{100} + 0.6e^{i\pi/3}\psi_{311}$$

where the $\psi_{n\ell m}$ denote normalized energy eigenstates. If $\left(\hat{L}_x, \hat{L}_y, \hat{L}_z\right)$ are the components of the orbital angular momentum operator, the expectation value of $\hat{L}_x^2$ in this system is

(a) $1.5\hbar^2$ (b) $0.18\hbar^2$ (c) $0.36\hbar^2$ (d) Zero

Ans. : (b)

Solution. :

The wave function of electron in a given system of hydrogen atom is defined as

$$\psi(r,\theta,\phi) = 0.8\,\psi_{100} + 0.6e^{i\pi/3}\psi_{3,1} = 0.8|1,0,0\rangle + 0.6e^{i\pi/3}|3,1,1\rangle$$

The expectation value of $\hat{L}_x^2$ is given by $\hat{L}_x^2 = \frac{1}{2}\left[L^2 - L_2^2\right]$

Let us determine the expectation value of $\langle L^2\rangle$

$$L^2|\psi\rangle = L^2\left(0.8|1,0,0\rangle + 0.6e^{i\pi/3}|311\rangle\right) = 0.8L^2|1,0,0\rangle + 0.6e^{i\pi/3}L^2|311\rangle$$

$$= 0.8\hbar^2\ 0(0+1)|1.0,0\rangle + 0.8e^{i\pi/3}2\hbar^2|311\rangle = 0.6e^{i\pi/3}2\hbar^2|311\rangle$$

or $\left\langle \psi \left| L^2 \right| \psi \right\rangle = 036 \times 2\hbar^2 \langle 311|311\rangle = 0.72\hbar^2$

Similarly the expectation value of $\left\langle L_z^2 \right\rangle$ is

$L_z|\psi\rangle = L_z\left(0.8|1,0,0\rangle + 0.6e^{i\pi/3}L_z|3,1,1\rangle\right) = 0.8(0.\hbar)|1,0,0\rangle + 0.6e^{i\pi/3}\hbar|311\rangle$

$L_z^2|\psi\rangle = 0.6e^{i\pi/3}\hbar^2|311\rangle$

or , $\left\langle \psi \left| L_2^2 \right| \psi \right\rangle = 0.36\hbar^2 \langle 311|311\rangle = 0.36\hbar^2$

The expectation value of L_x^2 is given by

$$\left\langle L_x^2 \right\rangle = \left\langle \psi \left| \frac{1}{2}\left(L^2 - L_z^2\right| \psi \right\rangle = \frac{1}{2}\left(\left\langle \psi \left| L^2 \right| \psi \right\rangle - (\psi) L_z^2 |\psi\rangle\right) = \frac{1}{2}\left(0.72\hbar^2 - 0.36\hbar^2\right) = 0.18\hbar^2$$

SECTION B-(Only for Ph.D. candidates)

Q6. A one-dimensional quantum harmonic oscillator of natural frequency ω is in thermal equilibrium with a heat bath at temperature T. The mean value $\langle E \rangle$ of the energy of the oscillator can be written as

(a) $\frac{\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2k_BT}\right)$ (b) $\frac{\hbar\omega}{2}\text{csch}\left(\frac{\hbar\omega}{2k_BT}\right)$

(c) $\frac{\hbar\omega}{2}\text{sech}\left(\frac{\hbar\omega}{2k_BT}\right)$ (d) $\frac{\hbar\omega}{2}\tanh\left(\frac{\hbar\omega}{2k_BT}\right)$

Ans. : (a)

Solution. :

Consider a one-dimensional quantum harmonic oscillator of natural frequency ω is in thermal equilibrium with a heat bath at temperature T. The energy of the oscillator

$$E = \sum_{i=1}^{N} \hbar\omega\left(n_i + \frac{1}{2}\right)$$

The single oscillator partition function is

$$Z_1 = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)} = e^{-\beta\hbar\omega/2}\sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega/2} - e^{-\frac{\beta\hbar\omega}{2}}} = \frac{1}{2\sinh h\beta\hbar/2}$$

The partition function for the system is

$$Z_N = (Z_1)^N = \left(\frac{1}{2\sinh\beta\hbar\omega/2}\right)^N$$

This yields for the partition function

$$\ln Z_N = N\ln Z_1 = N\left(\ln\frac{1}{2} - \ln\sinh\frac{\beta\hbar\omega}{2}\right)$$

Which enables us to determine the internal energy

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z_N \quad \Rightarrow \langle E\rangle = -\frac{\partial}{\partial\beta}N\left(\ln\frac{1}{2} - \ln\sinh\left(\frac{\beta\hbar\omega}{2}\right)\right]$$

$$\Rightarrow \langle E\rangle = N\frac{\hbar\omega}{2}\frac{\cos h\left(\beta\frac{\hbar\omega}{2}\right)}{\sin h\left(\beta\frac{\hbar\omega}{2}\right)} = N\frac{\hbar\omega}{2}\cot h\left(\beta\frac{\hbar\omega}{2}\right)$$

The mean value $\langle E\rangle / N$ of the energy of the oscillator can be written as

$$\frac{\langle E\rangle}{N} = \frac{\hbar\omega}{2}\cot h\left(\frac{\hbar\omega}{2k_BT}\right)$$

Q7. A quantum mechanical system which has stationary states $|1\rangle, |2\rangle$ and $|3\rangle$, corresponding to energy levels $0\,eV, 1\,eV$ and $2\,eV$ respectively, is perturbed by a potential of the form

$$\hat{V} = \varepsilon|1\rangle\langle 3| + \varepsilon|3\rangle\langle 1|$$

where, in $eV, 0 < \varepsilon \ll 1$.

The new ground state, correct to order ε, is approximately.

(a) $\left(1-\frac{\varepsilon}{2}\right)|1\rangle + \frac{\varepsilon}{2}|3\rangle$

(b) $|1\rangle + \frac{\varepsilon}{2}|2\rangle - \varepsilon|3\rangle$

(c) $|1\rangle - \frac{\varepsilon}{2}|3\rangle$

(d) $|1\rangle + \frac{\varepsilon}{2}|3\rangle$

Ans. : (c)

Solution. :

The correction for wave function is given by $|n\rangle_{new} = |n^0\rangle + \underset{k+n}{E}|k^0\rangle \frac{\langle n|v|k\rangle}{E_n^{(0)} - E_k^{(0)}}$

The correction in the ground state is $|1\rangle_{new} = |1\rangle + |2\rangle\frac{\langle 1|v|2\rangle}{E_1^{(0)} - E_2^{(0)}} + |3\rangle\frac{\langle 1|v|3\rangle}{E_1^{(0)} - E_3^{(0)}}$

Let us evaluate the value of $\langle 1|v|2\rangle$ i.e.

$$\langle 1|v|2\rangle = \langle 1|\left(3|1\rangle\langle 3| + e|3\rangle\langle 1|\right)|2\rangle = \ell\langle 1|1\rangle\langle 3|2\rangle + \ell\langle 1|3\rangle\langle 1|2\rangle = 0$$

Similarly, Let us evaluate the value of $\langle 1|v|3\rangle$

$$\langle 1|v|3\rangle = \langle 1|\left(\ell|1\rangle\langle 3| + \ell|3\rangle\langle 1|\right)|3\rangle = \ell\langle 1|1\rangle\langle 3|3\rangle + \ell\langle 1|3\rangle\langle 1|3\rangle = \ell$$

Substituting the value of $\langle 1|v|2\rangle$ and $\langle 1|v|3\rangle$ in the equation for wave function we get

$$|1\rangle_{\text{new}} = |1\rangle + \frac{|2\rangle.0}{(0-1)} + |3\rangle\frac{\ell}{0-2} \quad \Rightarrow |1\rangle_{\text{new}} = |1\rangle - \frac{\ell}{2}|3\rangle$$

Q8. A particle of mass m, confined to one dimension x, is in the ground state of a harmonic oscillator potential with a normalized wave function

$$\psi_0(x)=\left(\frac{2a}{\pi}\right)^{\frac{1}{4}}e^{-ax^2}$$

where $a=\dfrac{m\omega}{2\hbar}$. Find the expectation value of x^8 in terms of the parameter a

Ans. : $\dfrac{105}{256a^4}$

Solution. :

The ground state wave function for a particle of must in confined in one dimensional harmonic potential is given by $\psi_0(x)=\left(\frac{2a}{\pi}\right)^{1/4}e^{-ax^2}$

The expectation value of x^8 is given by $\langle x^8\rangle=\langle\psi|x^8|\psi\rangle=\left(\frac{2a}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}x^8e^{-2ax^2}dx$

Introducing new variable, $2ax^2=t\Rightarrow x=\frac{1}{\sqrt{2a}}t^{1/2}\Rightarrow dx=\frac{1}{\sqrt{2a}}\frac{1}{2}t^{-1/2}dt$

Substituting the value of x in above equation, we get

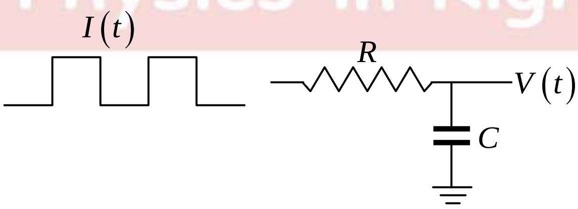
$$\langle x^8\rangle=\left(\frac{2a}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}x^8e^{-2ax^2}dx=\left(\frac{2a}{\pi}\right)^{1/2}.\left(\frac{1}{2a}\right)^4.\frac{1}{2(2a)^{1/2}}.2.\int_0^{\infty}t^{7/2}e^{-t}dt$$

$$=\frac{1}{\sqrt{\pi}}\frac{1}{(2a)^4}\int_0^{\infty}t^{9/2}e^{-t}dt=\frac{1}{\sqrt{\pi}}\frac{1}{(2a)^4}\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}=\frac{105}{256a^4}$$

TIFR-2017 (Electronics Question and Solution)

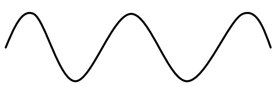
SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. A current source produces a square wave $I(t)$ of $1.0V$ peak-to-peak voltage and is used to drive the RC circuit shown below.

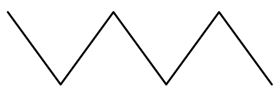


Which of the following represents the correct voltage across the capacitor C?

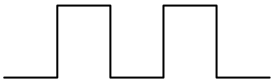
(a)



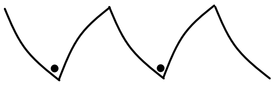
(b)



(c)



(d)

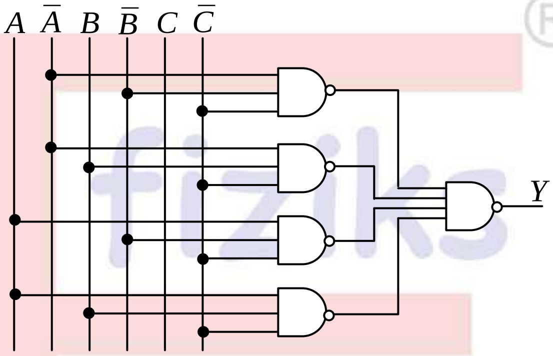


Ans. : (b)

Solution. :

RC circuit behaves as an integrator circuit.

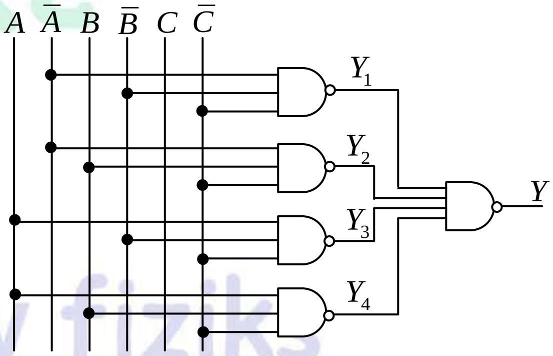
Q2. The output (Y) of the following circuit will be



(a) $\overline{C}$ (b) $\overline{B}$ (c) $\overline{A}$ (d) $\overline{A}+B+\overline{C}$

Ans. : (a)

Solution. :



$$Y = \overline{Y_1.Y_2.Y_3.Y_4} = \overline{Y}_1 + \overline{Y}_2 + \overline{Y}_3 + \overline{Y}_4$$

$$Y_1 = \overline{\overline{A}\,\overline{B}\,\overline{C}},\quad Y_2 = \overline{\overline{A}B\overline{C}},\quad Y_3 = \overline{A\overline{B}\,\overline{C}},\quad Y_4 = \overline{AB\overline{C}}$$

$$\Rightarrow Y = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}B\overline{C} + A\overline{B}\,\overline{C} + AB\overline{C}$$

$$\Rightarrow Y = \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + \overline{A}B\overline{C} + AB\overline{C}$$

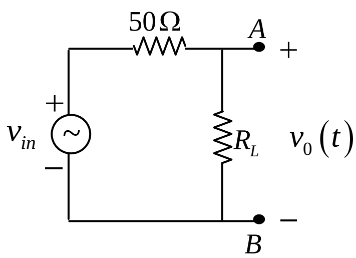
$$\Rightarrow Y = \left(\overline{A}+A\right)\overline{B}\,\overline{C} + \left(\overline{A}+A\right)B\overline{C}$$

$$\Rightarrow Y = \overline{B}\,\overline{C} + B\overline{C} = \left(\overline{B}+B\right)\overline{C} = \overline{C}$$

Q3. An AC voltage source has an internal resistance of 50Ω and is specified to deliver an rms voltage of $50V$ to a matched load. If you connect this AC source to a cathode- ray oscilloscope with $1M\Omega$ input setting, what will be the peak-to-peak voltage you observe?

Ans. : 283

Solution. :



If $R_L = 50\Omega$, $v_0 = 50V\ rms$, then $v_{in} = 100V\ rms$.

If $R_L = 1M\,\Omega$, $v_0 = \dfrac{10^6\,\Omega}{50\,\Omega + 10^6\,\Omega} \times 100V\ rms = 100V\ rms$

Peak voltage of input is $V_{0m} = 100\sqrt{2}\ V$.

Thus peak to peak voltage of input is $2V_{0m} = 200\sqrt{2}\ V = 283\ V$.

SECTION B- (only for Int.-Ph.D. candidates)

Q4. For exact calculation and minimum complexity, two four-digit binary numbers can be added with

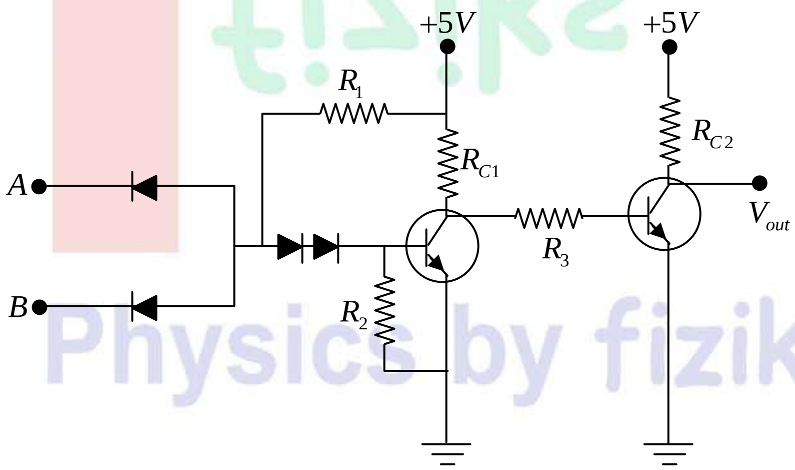
(a) 3 full adders and 1 half-adder (b) 2 full adders and 2 half-adders

(c) 1 full adder and 3 half-adders (d) 4 full adders

Ans. : (a)

Solution. :

Q5. Which digital logic gate is mimicked by the following silicon diode and silicon transistor circuit?

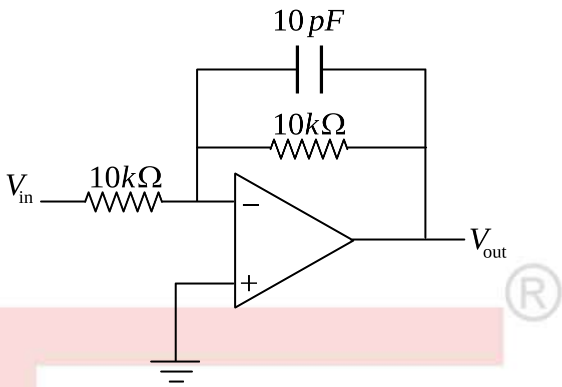


Solution:

AND Gate

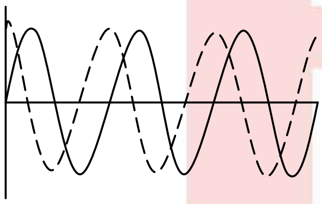
SECTION B-(Only for Ph.D. candidates)

Q6. The following circuit is fed with an input sine wave of frequency $50\,Hz$.

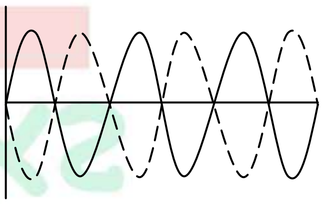


Which of the following graphs (solid line is input and dashed line is output) best represents the correct situation?

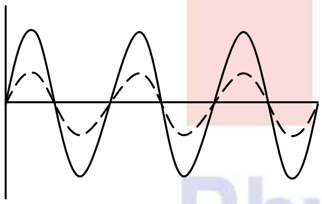
(a)



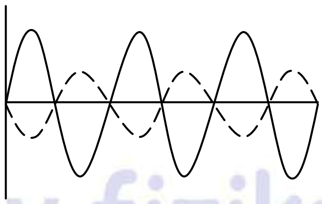
(b)



(c)



(d)



Ans. : (c)

Solution. :

$$\frac{v_0}{v_{in}} = -\frac{X_C \parallel R_F}{R_1} = -\frac{X_C R_F}{R_1\left(X_C + R_F\right)} = -\frac{R_F}{R_1\left(1 + R_F / X_C\right)} = -\frac{R_F}{R_1\left(1 + j\omega C_F R_F\right)}$$

$$\left|\frac{v_0}{v_{in}}\right| = \frac{\left(R_F / R_1\right)e^{j\pi}}{\sqrt{1+\left(2\pi fC_F R_F\right)^2}\,e^{j\theta}} = \frac{\left(R_F / R_1\right)}{\sqrt{1+\left(2\pi fC_F R_F\right)^2}}e^{j(\pi-\theta)} \text{ where } \theta = \tan^{-1}\left(2\pi fC_F R_F\right).$$

Let us calculate $2\pi fC_F R_F = 2\times 3.14\times 50\times\left(10\times10^{-12}F\right)\times\left(10\times10^{3}\Omega\right) = 314\times10^{-7} \to 0$.

$$\Rightarrow \left|\frac{v_0}{v_{in}}\right| = \frac{\left(R_F / R_1\right)}{\sqrt{1+\left(2\pi fC_F R_F\right)^2}} = \frac{(1)}{\sqrt{1+(0)^2}} = 1 \quad \Rightarrow |v_0| = |v_{in}|$$

and $\theta = \tan^{-1}\left(2\pi fC_F R_F\right) = \tan^{-1}(0) = 0 \Rightarrow \phi = \pi$

TIFR-2017 (Atomic- Molecular Physics Questions and Solution)

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. The separation between neighbouring absorption lines in a pure rotational spectrum of the hydrogen bromide (HBr) molecule is $2.23\,meV$. If this molecule is considered as a rigid rotor and the atomic mass number of Br is 80, the corresponding absorption line separation in deuterium bromide (DBr) molecule, in units of meV, would be

(a) 2.234 (b) 1.115 (c) 4.461 (d) 1.128

Ans.: (d)

Solution:

$$\varepsilon_J = BJ(J+1)\ m^{-1},\quad \varepsilon_{J+1}-\varepsilon_J = 2B(J+1)\ m^{-1}$$

$$E_{J+1}-E_J = 2Bhc(J+1)\ \text{Joule}$$

$$B = \frac{h}{8\pi^2 Ie} = \frac{h}{8\pi^2 \mu r^2 e}$$

$$\mu_{HBr} = \frac{m_H m_{Br}}{m_H + m_{Br}} = \frac{1\times 80}{1+80}\times 1.66\times 10^{-27}\,kg$$

$$\mu_{DBr} = \frac{m_D m_{Br}}{m_D + m_{Br}} = \frac{2\times 80}{2+80}\times 1.66\times 10^{-27}\,kg$$

$$\frac{(E_{J+1}-E_J)_{DBr}}{(E_{J+1}-E_J)_{HBr}} = \frac{\mu_{HBr}}{\mu_{DBr}} = \frac{80}{81}\times\frac{82}{160} = \frac{41}{81}$$

$$(E_{J+1}-E_J)_{DBr} = \frac{41}{81}\times 2.23 = 1.128\ meV$$

Q2. The energy of an electron in the ground state of the He atom is $-79\,eV$. Considering the Bohr model of the atom, what would be 10 times the first ionization potential for a He^+ ion, in units of eV?

Ans.: 246

Solution:

Ground state energy of He atom $E_{He}^{Gs} = -79\,eV$

He - atom becomes He^+ ion after ionization.

Ground state energy of He^+ is

$$E_{He^+}^{GS} = -13.6\frac{Z^2}{n^2} = -13.6\times 4 \qquad \Rightarrow E_{He^+}^{GS} = -54.4\,eV$$

First ionization potential (IP) of $He^+ = E_{He^+}^{GS} - E_{He}^{GS} = -54.4-(-79) = 24.6\,eV$

So, ten times of first $\text{I.P} = 10\times 24.6 = 246\ eV$

SECTION B-(Only for Ph.D. candidates)

Q3. Hydrogen atoms in the atmosphere of a star are in thermal equilibrium, with an average kinetic energy of $1eV$. The ratio of the number of hydrogen atoms in the 2nd excited state $(n=3)$ to the number in the ground state $(n=1)$ is

(a) 5.62×10^{-6} (b) 3.16×10^{-11}

(c) 3.16×10^{-8} (d) 1.33×10^{-8}

Ans.: (d)

Solution:

Average kinetic energy $=\frac{3}{2}kT=1eV \qquad \Rightarrow kT=\frac{2}{3}eV$

Energy of ground state $E_1=-13.6eV$

Energy of second excited state $(n=3)$; $E_3=-\frac{13.6}{3^2}=-\frac{13.6}{9}eV$

$$\frac{N_3}{N_1}=e^{-(E_3-E_1)/kT}=e^{\left(+\frac{13.6}{9}-13.6\right)/(2/3)}=e^{-\frac{3}{2}\times\frac{8}{9}\times13.6}=e^{-54.4/3}\Rightarrow\frac{N_3}{N_1}=1.33\times10^{-8}$$

TIFR-2017 (Solid State Physics Questions and Solution)

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. Consider a $2-D$ square lattice. The ratio of the kinetic energy of a free electron at a corner of the first Brillouin zone (E_c) to that of an electron at the midpoint of a side face-of the same zone (E_m) is $\frac{E_c}{E_m}=$

(a) 2 (b) $\sqrt{2}$ (c) 1 (d) $\frac{1}{2}$

Ans: (a)

Solution:

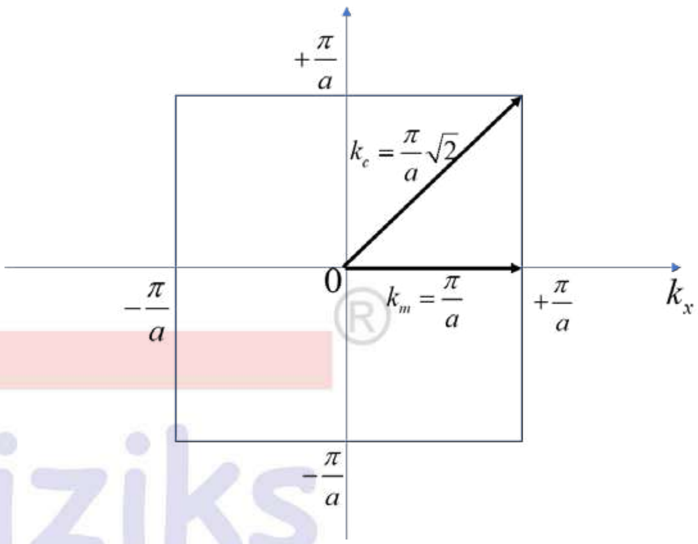
Kinetic energy of the free electron is written as a function of k is $E=\frac{\hbar^2k^2}{2m}$

Kinetic energy of the free electron at a corner of the first Brillouin zone (E_c) is

$$E_c=\frac{\hbar^2k_c^2}{2m}=\frac{\hbar^2}{2m}\frac{2\pi^2}{a^2}$$

Kinetic energy of the free electron at the midpoint of a side face-of the same zone (E_m) is

$$E_c = \frac{\hbar^2 k_c^2}{2m} = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}$$



The ratio is

$$\frac{E_c}{E_m} = \frac{\frac{\hbar^2}{2m}\frac{2\pi^2}{a^2}}{\frac{\hbar^2}{2m}\frac{\pi^2}{a^2}} = 2$$

Thus, the correction answer is option (a)

SECTION B- (only for Int.-Ph.D. candidates)

Q2. In two dimensions, two metals A and B, ha the number density of free electrons in the ratio

$$n_A : n_B = 1:2$$

The ratio of their Fermi energies is

(a) $2:3$ (b) $1:8$ (c) $1:2$ (d) $1:4$

Ans: (c)

Solution:

A d-dimensional metal with volume L^d contain N electrons and can be written as

$$E_F = \frac{\hbar^2}{2mL^2}(N\zeta_d)^{2/d}$$

For $d=1,\ \zeta_d = \frac{\pi}{2};\quad d=2,\ \zeta_d = \pi;$ and $d=3,\ \zeta_d = 3\pi^2$

For two dimensions, the Fermi energy is

$$E_F = \frac{\hbar^2}{2mL^2}(N\pi)^{2/2} = \frac{\hbar^2}{2m}\frac{N\pi}{L^2} = \frac{\pi\hbar^2}{2m}n$$

The ratio of the Fermi energy of two metals A and B is

$$\frac{E_{FA}}{E_{FB}} = \frac{n_A}{n_B} = \frac{1}{2}$$

Thus, the correct answer is option (c).

SECTION B-(Only for Ph.D. candidates)

Q3. Electrons in a metal are scattered by both impurities and phonons. The impurity scattering time is $8\times10^{-12}\,s$ and the phonon scattering time is $2\times10^{-12}\,s$. Taking the density of electrons to be $3\times10^{14}\,m^{-3}$, find the conductivity of the metal in units of $AV^{-1}m^{-1}$. [Assume that the effective mass of the electrons is the same as that of a free electron.]

Ans: 1.4×10^{-5}

Solution:

Conductivity (σ) and hence the resistivity ($\rho=\frac{1}{\sigma}$)of the metal is

$$\sigma=\frac{ne^2\tau}{m^*},\quad \rho=\frac{1}{\sigma}=\frac{m^*}{ne^2\tau},$$

Where, n is the electron concentration, m^* is the effective mass of the electron that is equal to the rest mass of the electron and τ is the collision time.

According to the Matthiessen rule the total resistivity ρ total can be approximated by adding up several different terms:

$$\rho=\rho_{phonons}+\rho_{impurity}=\frac{m}{ne^2\tau_{phonon}}+\frac{m}{ne^2\tau_{impurity}}=\frac{m}{ne^2}\left(\frac{1}{\tau_{phonon}}+\frac{1}{\tau_{impurity}}\right)$$

Given, $\tau_{phonon}=8\times10^{-12}\,s$, $\tau_{impurity}=2\times10^{-12}\,s$, $n=3\times10^{14}\,m^{-3}$ and $m=9.1\times10^{-31}\,kg$

$$\rho=\frac{9.1\times10^{-31}}{3\times10^{14}\times\left(1.6\times10^{-19}\right)^2}\left(\frac{1}{8\times10^{-12}}+\frac{1}{2\times10^{-12}}\right)=7.4\times10^4\,\Omega-m$$

The electrical conductivity is

$$\sigma=\frac{1}{\rho}=\frac{1}{7.4\times10^4}=0.135\times10^{-4}\,AV^{-1}m^{-1}=1.4\times10^{-5}\,AV^{-1}m^{-1}$$

TIFR-2017 (Nuclear Physics Questions and Solution)

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

Q1. Cosmic ray muons, which decay spontaneously with proper lifetime $2.2\,\mu s$, are produced in the atmosphere, at a height of $5km$ above sea level. These move straight downwards at 98% of the speed of light. Find the percent ratio $100\times\left(\frac{N_A}{N_B}\right)$ of the number of muons measured at the top of two mountains A and B, which are at heights $4,848\,m$ and $2,682\,m$ respectively above mean sea level.

Ans.: 52

Solution.:

Proper life time $\tau_0 = 2.2\times10^{-6}s$

Life time measured from the earth $\tau = \dfrac{\tau_0}{\sqrt{1-\frac{v^2}{c^2}}} = \dfrac{2.2\times10^{-6}}{\sqrt{1-(0.98)^2}} = 11.06\times10^{-6}s$

Time taken by muon to reach at the top of mountain A_1, $t_A = \dfrac{(5000-4848)}{0.98c} = \dfrac{152}{0.98c}$

$$N_A = N_0 e^{-\lambda t_A} = N_0 e^{-t_A/\tau} \Rightarrow N_A = N_0 e^{-\frac{152}{0.98c\times11.06\times10^{-6}}} \Rightarrow N_A = N_0 e^{-\frac{152}{3251}}$$

Similarly, $N_B = N_0 e^{-\frac{2318}{0.98c\times11.06\times10^{-6}}} \Rightarrow N_B = N_0 e^{-\frac{2318}{3251}}$

$$\Rightarrow \frac{N_A}{N_B} = \frac{N_0 e^{-\frac{152}{3251}}}{N_0 e^{-\frac{2318}{3251}}} \Rightarrow \frac{N_A}{N_B} = e^{0.67} = 1.95 \Rightarrow 100\times\left(\frac{N_A}{N_B}\right) = 195$$

$$\Rightarrow \frac{N_B}{N_A} = \frac{1}{1.95} = 0.52 \quad \Rightarrow 100\times\left(\frac{N_B}{N_A}\right) = 52$$

SECTION B- (only for Int.-Ph.D. candidates)

Q2. A deuteron of mass M and binding energy B is struck by a gamma ray photon of energy E_γ and is observed to disintegrate into a neutron and a proton. If $B \ll Mc^2$, the minimum value of E_γ must be

(a) $2B + \dfrac{B^2}{2Mc^2}$ (b) $\dfrac{1}{2}\left(3B + \dfrac{B^2}{Mc^2}\right)$

(c) $B + \dfrac{B^2}{Mc^2}$ (d) $\dfrac{1}{2}\left(2B + \dfrac{B^2}{Mc^2}\right)$

Ans.: (d)

Solution: $D + \gamma \rightarrow p + n$;

Momentum of photon $= \dfrac{E_\gamma}{c}$, Combined momentum of $(p+n) = \dfrac{E_\gamma}{c}$

Q-value of the nuclear reaction,

$Q = Mc^2 - (m_p + m_n)c^2 = 0 - B \quad \Rightarrow B = (m_p + m_n)c^2 - Mc^2$

Apply the conservation of energy $E_\gamma + Mc^2 = \sqrt{(m_p+m_n)^2 c^4 + \left(\dfrac{E_\gamma}{c}\right)^2 c^2}$

$$E_\gamma^2 + M^2c^4 + 2E_\gamma.Mc^2 = B^2 + 2BMc^2 + M^2c^4 + E_\gamma^2 \quad \Rightarrow 2E_\gamma Mc^2 = B^2 + 2BMc^2$$

$$\Rightarrow E_\gamma = \frac{B^2}{2Mc^2} + B \quad \Rightarrow E_\gamma = \frac{1}{2}\left(\frac{B^2}{Mc^2} + 2B\right)$$

Revised Edition-2022
H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

SECTION B-(Only for Ph.D. candidates)

Q2. A subatomic particle ψ and its excited state ψ^* have rest masses $3.1 GeV/c^2$ and $3.7 GeV/c^2$ respectively. A table of its assigned quantum numbers is given below.

Angular Momentum	Parity	C-Parity	Isospin	Electric charge
$J=1$	$P=-1$	$C=-1$	$I=0$	$Q=0$

If π^{0*} is an excited state of π^0 with a mass of about $1.3 GeV/c^2$, which of the following reactions is possible when the above quantum numbers are conserved?

(a) $\psi^* \to \gamma\gamma$

(b) $\psi^* \to \pi^0\pi^0$

(c) $\psi^* \to \psi\,\pi^+\pi^-$

(d) $\psi^* \to \psi\,\pi^{0*}$

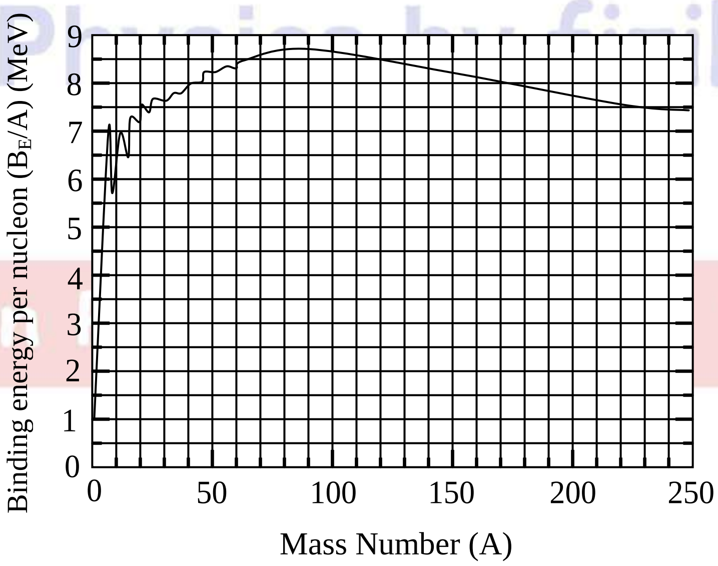
Ans.: (c)

Solution:

Bosons and their antiparticles have same parity.

	$\psi \to$	ψ +	π^+ +	π^-	
S	1	1	0	0	conserved
parity	-1	-1	-1	-1	conserved
Isospin	0	0	0	0	conserved
Q	0	0	$+1$	-1	conserved

Q3. In a theoretical model of the nucleus, the binding energy per nucleon was predicted as shown in the figure below



If a nucleus of mass number $A = 240$ undergoes a symmetric fission to two daughter nuclei each of mass number $A = 120$, write down the amount of energy released in this process, in units of MeV, using this theoretical model.

Ans.: **240**

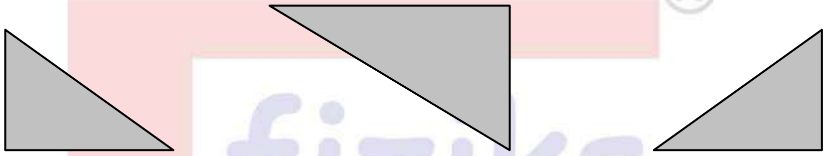
Solution: $^{240}X \rightarrow {}^{120}Y + {}^{120}Y$

$\Delta E = 2BE_Y - BE_X = (2\times120\times8.5)-(240\times7.5) = 240\times(8.5-7.5) = 240\ MeV$

General Physics

SECTION A-(For both Int. Ph.D. and Ph.D. candidates)

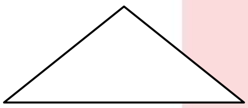
Q3. A solid tetrahedron (solid with four plane sides) has the following projections (drawn to scale) when seen from three different sides:



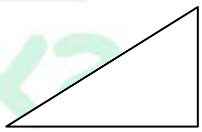
Left Top Right

When viewed from the front, its projection will be

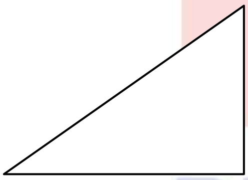
(a)



(b)



(c)

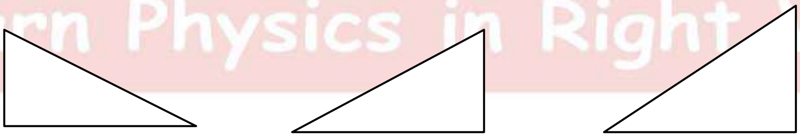


(d)

Ans: (b)

Solution:

The projection seen from front will have one of its side non-modulated and will be equal in length to one of sides of the left or right view. Only option which fits into this scheme is choice (b). Hence choice is (b)



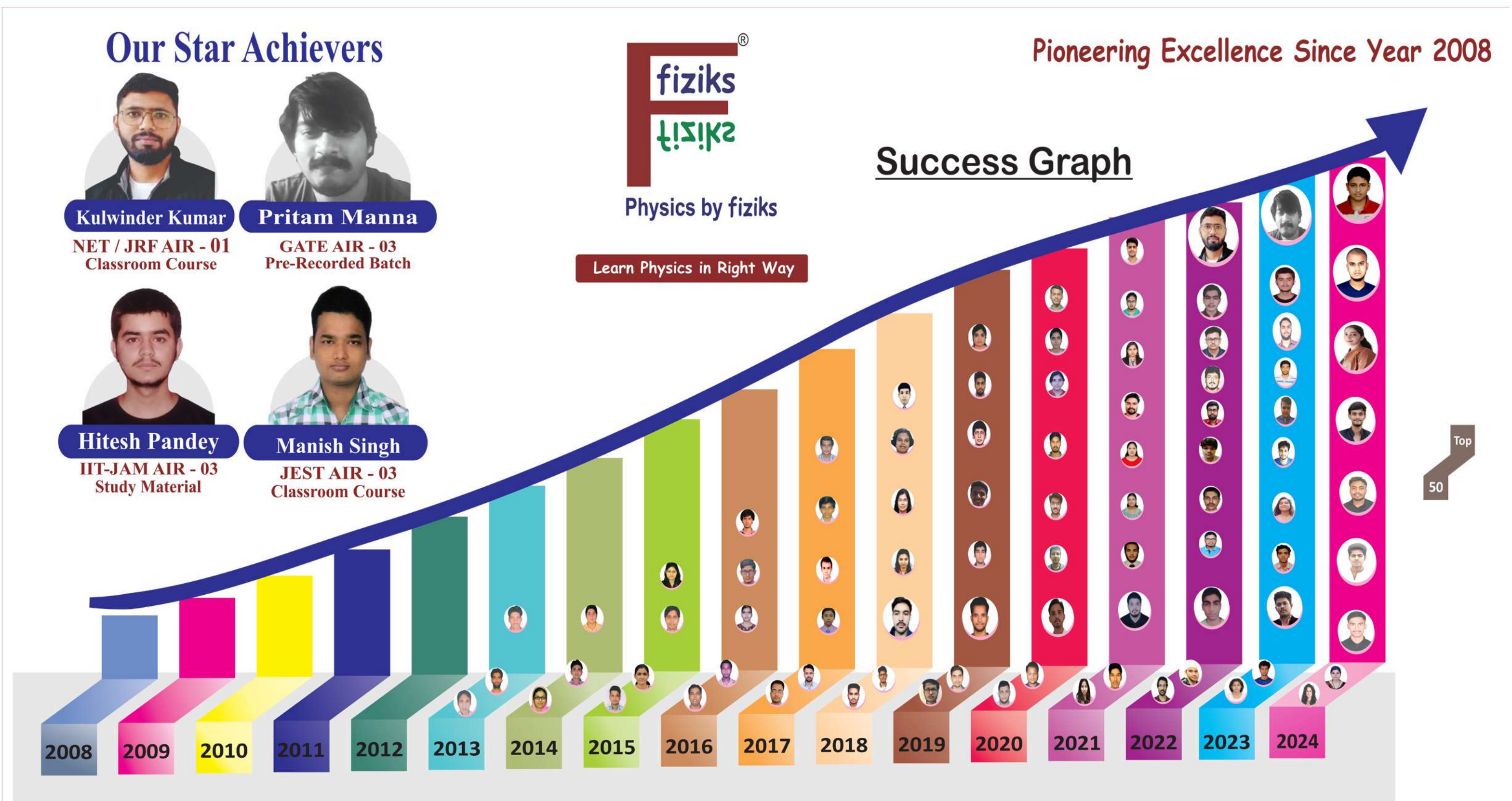
Left Right Front

H.No. 40-D, G.F., Jia Sarai, Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498, Website: www.physicsbyfiziks.com, Email: fiziks.physics@gmail.com

Physics by fiziks

Pioneering Excellence Since Year 2008

Learn Physics in Right Way



011-26865455, +9871145498

Head Office: House No. 40-D, Ground Floor, Jia Sarai Near IIT-Delhi, Hauz Khas, New Delhi-110016